Shedding and interaction of solitons in weakly disordered optical fibers

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Propagation of a soliton pattern through an optical fiber with weakly disordered dispersion is considered. Solitons, perturbed by this disorder, radiate, and, as a consequence, decay. The average radiation profile is found. The emergence of a long-range intra-channel interaction between the solitons, mediated by the radiation, is reported. We show that soliton in a multi-soliton pattern experience a random jitter: average force acting on a soliton is negligible and fluctuations of the soliton velocity are Gaussian, with a typical fluctuation estimated by $Dz^2\sqrt{\mu}$, where D measures the disorder strength, z is the distance passed by the soliton in the fiber and μ stands for the information rate (number of solitons per unit length of the fiber). We also present results of direct numerical simulation of the soliton decay and two-soliton interaction, confirming our theoretical analysis. Relevance of the results to fiber optics communication technology is discussed.

Introduction

Life is not perfect and fibers are not ideal. Production inability to achieve a 100% guaranteed control of the fiber parameters in the process of fiber pulling and preform manufacturing results in irregularities of the fiber structure. The structural disorder is built in the fiber. Effect of the disorder on the propagation and interaction of pulses, accumulates with propagation, i.e. the longer a pulse (pattern of pulses) travels along the fiber, the stronger disorder affects it. Even a weak disorder may cause an essential damage to the pulse (sequence of pulses) integrity. A strong effect of weak disorder in fiber dispersion coefficient on the shedding and interaction of pulses, problem which is crucial for progress in modern nonlinear fiber optics and related communication technology, is described in the paper.

In fiber optics communication pulse is used as a bit of information. For an ideal fiber, working in the regime of nonlinear transmission, a pulse of electric field is described by a stationary solution (soliton) of self-focusing Nonlinear Shrödinger equation (NLSE) with constant coefficients. Stationarity, in particular, means that soliton propagates through the fiber with a constant speed. (See the book [1] for detailed derivation of NLSE from Maxwell equations in a very general fiber optics setup.) The stationary soliton is a result of a fine balance between the fiber dispersion and nonlinearity [2–4].

A sequence of pulses launched into fiber forms a pattern, which codes transmitted message. Ideally, a pattern, carrying information, is a sequence of solitons, each positioned in the center of a slot allocated for the information bit. State "1" is assigned to a slot, if the soliton is present there and the state of the slot is "0" if the slot contains no soliton. The disorder, built in the fiber, breaks the ideal picture. (See also [5, 6] for a sample of other corrections to ideal NLSE important in fiber optics communications.) In the present manuscript we detail dynamics of single- and multi-soliton patterns in the presence of weak disorder in dispersion coefficient. Some preliminary results of the study, detailed and corrected here, were briefly described before in [7].

A soliton, propagating through the disordered system, shades radiation and, consequently, loses energy. However, in the case of weak disorder (weakness of disorder is actually required for successful fiber performance) destruction of the soliton is slow, i.e. its adiabatic description is possible. This implies separation of dynamical degrees of freedom into slow and fast modes. (See [8–10] for the general description of adiabatic perturbation approach to partial differential equations and [11, 12] for the application of the general method to various regular perturbations about the soliton solution of $1d$ NLSE.) The slow modes describe the soliton evolution and the fast ones correspond to the radiation. The soliton keeps its ideal shape (so that, at each instant, it is close to a stationary solution of the noiseless NLSE) with the soliton parameters (position, width, phase and phase velocity) evolving slowly. Waves shed by a soliton are moving away from it. The average intensity of the radiation (at sufficiently large t, $t < z$) is estimated as $D\eta^4 \ln(z/t)$. Here, η is the soliton amplitude, D measures the intensity of the disorder (which is assumed to be weak, $D \ll 1$), z stands for the distance passed by the soliton, and t is the retarded time, i.e. time counted from the moment when soliton was at the given position, z. (All the quantities are measured in the respective soliton units: the time unit is the soliton width, and the length unit corresponds to the distance passed by soliton during one turn on 2π of the soliton phase.) The radiation front runs out of the soliton (in t) with the velocity, which is $O(1)$. The foreruner of the front (which is the domain of $|t| \gg z$) decays exponentially with t/z. One finds that at any, however large z, the radiation in an immediate vicinity of the soliton is much less intense than the soliton itself, i.e. the soliton is always distinguishable from the radiation. Since the soliton losses its energy into radiation, its amplitude η decays with z. The degradation law is deterministic in spite of the original setting stochasticity. This is because the variation of η is determined by an integral over z, which is self-averaged quantity. The soliton degradation law, valid at any z, is

$$\eta = (1 + 32Dz/15)^{-1/4} . \qquad (0.1)$$

(A quantitative definition of the noise intensity D is given in the next section.) Notice, that the degradation of the soliton amplitude in the presence of disorder in dispersion coefficient was previously considered in [13], where estimations consistent with the analytic expression (0.1) were derived. Eq. (0.1) shows that the soliton starts to degrade essentially at $z \sim z_{degr} = 1/D$.

Next, we examine interaction of solitons at $1 \ll z \ll 1/D$ (when the soliton amplitude decrease is negligible) via the radiation shed under the action of disorder in dispersion. We show that the interaction is extremely long-range, due to the $1d$ nature of the system and also reflectiveless feature of the radiation. At any given z all solitons separated from the given one by $|t| \lesssim z$ act on this soliton with a force,. We find that the force is zero in average. Fluctuations of the force result in a Gaussian jitter of the soliton position. We find that for the two soliton case (i.e. for the pattern consisting of two solitons only so that no other solitons are present anywhere in the $|t| \lesssim z$ vicinity of the pair), fluctuations in their relative position, δy, are estimated by

$$\langle (\delta y)^2 \rangle \approx 0.37 \left[1 + \cos(2\alpha)\right] D^2 z^3 , \qquad (0.2)$$

where α is the phase mismatch of the solitons. Angular brackets in Eq. (0.2) (and below) mean averaging over realizations of disorder (i.e. over different fibers). In the general multi-soliton case fluctuations in the i-th soliton position are described by

$$\langle (\delta y_i)^2 \rangle \sim N D^2 z^3 , \qquad (0.3)$$

where N is the number of solitons in the same channel (propagating on a given frequency, i.e. with a given group velocity) in the $|t| \lesssim z$ vicinity of the pair. (To avoid a confusion, note, that effects of multi-channel interaction are not discussed here.) The interaction effect produces a displacement of a soliton of order unity (which, therefore, becomes dangerous as the soliton leaves the slot allocated for it and the information is lost) at $z \sim z_{int} = N^{-1/3} D^{-2/3}$. The interaction length, z_{int} is shorter than the degradation one, z_{degr}, so that our approximation is justified: solitons acquire significant shifts in their positions before any essential decrease of the soliton amplitude (or, generally, essential distortion of its shape) is observed.

The material in the paper is organized as follows. General fiber optics relations relevant to our analysis are presented in Section I. The single soliton results are detailed in Section II of the manuscript. In Section III we analyze peculiarities of the interaction of two solitons via the radiation emitted. In Section IV we discuss a generalization of the two-soliton interaction effects for the multi-soliton case. In Section V we present results of the direct numerical simulations for single-soliton and two-soliton cases. Section VI contains Conclusions. Details of calculations are given in Appendices.

I. BASIC RELATIONS

This Section is devoted to introduction into general problem of optical signal nonlinear propagation through an imperfect fiber. Basic equations governing propagation of a pulse through such a fiber are introduced in Section I A. Section I B is devoted to discussing parameters of real fibers, used in optics communication technology. Section I C introduces the formalism of a signal separation into localized modes (solitons) and delocalized modes (radiation). The general consequences of the weakness of disorder for the separation formalism are discussed in Section I D.

A. NLS with frozen disorder

Optical fibers are wave-guides based on the effect of complete internal reflection. A typical fiber consists of core with higher refractive index and of gliding with lower refractive index. Diameter of the fiber core corresponds to the first transverse mode at the carrier frequency of the signal. Therefore, light pulses can be described in terms of a single mode electro-magnetic field, propagating along the fiber. Then, the field can be treated as one-dimensional. Imperfectness of the fiber (disorder, built in the fiber) is mainly coming from variations in its diameter and chemical composite. Since a signal decays, amplifiers should be inserted in the fiber line, to maintain the signal's amplitude. Below, we discuss equations, averaged over the inter-amplifier distance, provided the attenuation is compensated by amplification.

A universal description of the signal envelope in the reference frame moving with the wave packet group velocity is given by NLSE (see, e.g., [1])

$$-i\partial_z \Psi = d(z)\partial_t^2 \Psi + 2|\Psi|^2 \Psi , \qquad (1.1)$$

explaining dynamics of electro-magnetic wave packet, with envelope $\Psi(z,t)$. The packet propagates in z (which is coordinate along the fiber), being subjected to dispersion in retarded time t (i.e. time counted from the moment when soliton passes through a given position, z) and to the Kerr nonlinearity. Eq. (1.1) assumes that fluctuations in the chromatic dispersion coefficient, $d(z)$, characterizing irregularity of the fiber, have a greater effect on propagation of pulses than fluctuations of any other coefficients there, say of the Kerr nonlinearity (which is, therefore, constant, re-scaled to 2 in the above equation). Eq. (1.1) is a result of the Maxwell equations averaging which accounts for geometrical features of the fiber core and gliding. Other averaging, also accounted for in (1.1), is performed over the amplifier spacing. Real-world problems in fiber-optics communication may require an account for corrections to Eq. (1.1), e.g. for subleading corrections coming from averaging over amplifier spacing [18]. We argue in Subsection I B that the extra terms produce only small, irrelevant corrections to the soliton interaction discussed in the paper. Eq. (1.1) also accounts for averaging over all the scales related to Polarization Mode Dispersion (PMD), which is the major effect in fiber optics communications associated with structural disorder. (PMD is caused by variations of ellipticity of the fiber, called birefringence.) Thus, in this manuscript we do not consider PMD, assuming that optical pulse is linearly polarized.

Only recently the chromatic dispersion profile, $d(z)$, became experimentally accessible. The high-precision measurements [19, 20] demonstrated a significance of the dispersion randomness. The chromatic dispersion in optical fibers comes from two sources. The first source is the medium itself. Material dispersion in modern fibers is a relatively stable parameter, uniformly distributed along the fiber. That is why we assume here, that the dispersion does not fluctuate in time. The second source is due to specific geometry of the wave-guide profile. Existing technology does not provide accurate control of the wave-guide geometry in fibers, so that the actual dependence of the dispersion coefficient on the wavelength is complicated. As a result, the typical magnitude, d_{var}, of random variations of fiber chromatic dispersion $d(z)$, can achieve, or in some cases even become greater than, that of the mean dispersion. A typical scale of the disorder variations, z_{var}, is much less than all relevant scales in the problem. (See, Section I B for discussion of real-world numbers and estimations.)

It is convenient to separate constant part of d (which we re-scale to unity) and its fluctuating part ξ: $d = 1+\xi$, where ξ is a random function of z, correlated on the scale z_{var}. We examine statistical properties of the fibers, which represent averaging over many realizations of the disorder $\xi(z)$ (over many fibers). Those objects allow to establish both typical fluctuations and probability of large deviations from the typical value for different quantities. Being interested in phenomena occurring on scales, larger than z_{var}, one can treat the disorder ξ as a short-correlated one. Then the first two cumulants of ξ are

$$\langle\xi\rangle = 0, \quad \langle\xi(z_1)\xi(z_2)\rangle = D\delta(z_1 - z_2)\,, \tag{1.2}$$

where $\langle\dots\rangle$ marks averaging over realizations of the disorder (over different fibers). The coefficient D (to be called noise intensity) is estimated as $D \sim z_{var}d_{var}^2$. High-order cumulants of ξ are negligible as containing higher powers of z_{var}. In other words, statistics of ξ is Gaussian. Also, the smallness of z_{var} (in comparison with relevant z-scales) leads to the inequality $D \ll 1$, which just means that the disorder is weak. The weakness of disorder is, actually, a necessary condition for a successful fiber performance.

Let us notice, that describing propagation of a signal, we adopt mixed optical-quantum mechanical notations and terminology. Indeed, the traditional optical notation, t, is reserved for retarded time, since, experimentally, the electro-magnetic field envelope is measured as a function of time, and also because t in Eq. (1.1) is a descendant of the real time in the original Maxwell equations, the equation was derived from. From the other side, the retarded time is proportional to real time minus position along the fiber z, over velocity of light, and, therefore, t is also carrying certain spatial sense. Besides, Eq. (1.1), called Nonlinear Schrödinger equation in direct analogy with the famous Linear Shrödinger equation, is a parabolic equation with second order derivative over time t, and not over the coordinate along the fiber z. The analogy with quantum mechanics is extremely helpful and will be used in later discussions and derivations. It explains why we treat t more like a spatial variable rather than a temporal one, marking oscillations in t by "wave vectors", which would be natural to call "frequencies" in a pure optical context. (To avoid a misunderstanding, let us stress, that the frequencies have no relation to the frequency of the original electromagnetic wave.)

Another remark is about relevance of the physics described by Eq. (1.1) for the phenomenon of localization of light in disordered medium [21]. As it was mentioned above, the disorder term ξ originates from fluctuations of the wave-guide dispersion, and not from the material component of the dispersion. Fluctuations of the material disorder was not accounted for in Eq. (1.1). Nevertheless, we find it useful to briefly discuss here its effect on propagation of light. The material disorder is associated with irregularities of the fiber core and gliding (impurities) on a very short, atomic scales. The light scattering on the impurities leads to the well known phenomenon of localization of light, taking place at the larger scale, usually called localization length [21]. The localization length is inversely proportional to the strength of the material disorder. The material used for manufacturing modern fibers is usually very clean, so that the localization length essentially exceeds the distance between filters, which, in the typical fiber lines, are placed at the amplifier stations. Filters cut the back scattering of light, and therefore, destroy coherence, required for emergence of the localization phenomenon. As a result, presence of a (very low intensity) material disorder does not play any significant role in fiber optics communications. Notice, also, that the scale of the wave-guide disorder variations, z_{var}, essentially exceeds the wave-length of light, that allows not to take into account the back-scattering of light due to the wave-guide disorder. (The later separation of scales allows to reduce the hyperbolic Maxwell equations to the parabolic equation (1.1) in the envelope approximation.) Therefore, no localization phenomena due to this disorder is possible. For the sake of generality, let us also note, that a role of the disorder in the context of the localization-delocalization transition was investigated for the non-linear Schrödinger equation (see, e.g., [22]). However, in the solid state physics frozen disorder means, that noise is z-independent (in our notations). The t-dependent noise is very different from the

z-dependent one, studied here, and to the best of our knowledge, the former case does not correspond to any situation of interest in fiber optics communications.

B. Real-world transmission parameters

The equation (1.1) is written in the dimensionless units, which are transformed from the real-world fiber units according to the following rules. The envelope of the electric field is in the form $E = \mathrm{Re}\left[\sqrt{P_0}\Psi e^{i\omega_0 t}\right]$, where P_0 is the peak pulse power and ω_0 is the carrying frequency of the signal. The propagation variable is $z = Z(\alpha_K P_0/2)$, where Z is distance along the fiber and α_K is the Kerr nonlinearity coefficient. The Kerr coefficient can be expressed in terms of other fiber parameters, $\alpha_K = 2\pi n_2/(\lambda S_{eff})$, where n_2 is the nonlinear component of fiber refractive index, λ is operating wavelength, and S_{eff} is an effective core area of the fiber. The other coordinate is $t = (T - Z/c)/\tau_0$, where T is time, c is the light velocity along the fiber ($T - Z/c$ is just the retarded time), and τ_0 is the pulse width. The dispersion coefficient is $d = 2\beta_2/(\alpha P_0 \tau_0^2)$, where β_2 is the second order dispersion parameter. We preset here the parameters' set for a standard example of dispersion shifted fiber: $\beta_2 = 0.1\mathrm{ps}^2/\mathrm{km}$, $\alpha = 2\mathrm{W}^{-1}\mathrm{km}^{-1}$, $\lambda = 1550\mathrm{nm}$, $\tau_0 = 7\mathrm{ps}$, $P_0 = 2\mathrm{mW}$.

The typical scale of the disorder variations, z_{var}, can be extracted from experimental measurements [19, 20], which show that z_{var}, is shorter than $\sim 1 - 2km$. Resolution of the experimental method is $1 - 2$ km, while one expects that the typical scale of the variations is, actually, one to two orders of magnitude shorter, $\sim 10 - 100$ m, i.e. it is fixed by the size of the production facility. In any case, z_{var} appears to be essentially shorter than the other relevant scales describing the long-haul transmission. It was also reported in [19] that fluctuations of the dispersion coefficient in a sample of the "dispersion shifted" fiber are of the order of its average value, i.e. $\delta\beta_2 \sim 0.5\mathrm{ps}^2/\mathrm{km}$. Therefore, for the pulse width of $\sim 7ps$ (that corresponds to 28 Gb/s single-channel transmission rate) and the nonlinear length, $z_{nl} = (\alpha P_0)^{-1} \sim$ 250km, the noise intensity $D = z_{var} d_{var}^2$ is estimated by $10^{-3} - 10^{-2}$. Then, the soliton interaction is seen at $z_{int} = 1/\sqrt{D} \sim 2,500 - 7,500\mathrm{km}$. Notice, however, that decrease of the pulse width by a factor q (correspondent to the factor q increase of the transmission rate) leads to the q^2 decrease of z_{int}.

Let us now discuss applicability criteria of the approximations leading to Eq. (1.1) for the real world situation in fiber-optics communication technology. An important additional scale in optical communication systems is imposed by fiber losses γ. Compensation of energy losses require use of in-line optical amplifiers separated by $z_{amp} \sim \gamma^{-1}$. The value of z_{amp} is usually $40 - 70$km. Soliton based optical communications is possible if dispersion length, $z_{disp} = \tau_0^2/\beta_2$, length of nonlinearity and amplification spacing are related in the following way $z_{disp} \sim z_{nl} \gg z_{amp}$. Oscillations of the pulse amplitude due to fiber losses in this case can be eliminated by averaging over z_{amp}, resulted in Eq. (1.1). Thus, respective corrections to Eq. (1.1) can be estimated as $(z_{amp}/z_{disp})^2$, as it is shown in [18]. For the above example the parameter, $(z_{amp}/z_{disp})^2$, is estimated as 10^{-2}, which can be of the same order of magnitude as noise strength D. Therefore, exclusion of the correction term from Eq. (1.1), as well as the validity of the averaging procedure over ξ, both require an additional justification. The correction term provides deterministic and stochastic contribution to optical pulse. Deterministic contribution does not produce an additional continuous radiation and provides only weak shape deformation of the optical soliton. The stochastic contribution is $(z_{disp}/z_{amp})^2$ times smaller than the main stochastic contribution considered in the paper. Therefore, averaging over the amplifier spacing does not change the major characteristic of the disorder D, and affects only the correlation length, i.e. the averaging changes the original z_{var} to z_{amp}. The latter scale is still much smaller than all other relevant scales, and, therefore, Eq. (1.1) does explain situation of practical interest for fiber optics communications.

C. Separation into localized-delocalized modes

One assumes that at the origin (fiber entrance), $z = 0$, the signal Ψ is close to N-soliton solution of the no-disorder NLSE, i.e. of Eq. (1.1) with $d = 1$. The disorder in the dispersion d, ξ, disturbs the ideal N-soliton pattern. Our task here is to describe evolution of Ψ under action of the disorder. The weakness of the disorder ξ and the localized nature of the initial profile $\Psi(0,t)$, suggest the following decomposition

$$\Psi = \Psi_{sol} + \Psi_{con}\,, \tag{1.3}$$

where Ψ_{sol} is the localized (soliton) part of the envelope and Ψ_{con} stands for radiation (de-localized part). If there is no disorder ($\xi = 0$) Ψ_{sol} is a solution of the $\xi = 0$ version of (1.1) and $\Psi_{con} = 0$. Therefore, Ψ_{con} is $O(\xi)$.

In the single-soliton case we have

$$\Psi_{sol} = \frac{\eta}{\cosh\left[\eta(t-y)\right]} \exp\left[i\varphi + i\beta(t-y)\right]\,, \tag{1.4}$$

where η, y, φ and β are amplitude, position, phase and phase velocity of the soliton. The disorder ξ drives a complicated z-dependence of the soliton parameters η, φ, β and y, whereas in absence of the disorder (at $\xi = 0$) η and β are z-independent, and y and φ are linear functions of z. We represent the soliton phase $\varphi(z)$ in the form

$$\varphi = \alpha + \int_0^z \mathrm{d}z' \eta^2(z')\,, \tag{1.5}$$

where α is a new parameter (which is z-independent in the absence of the disorder).

It is convenient to pass from the radiation field, Ψ_{con}, to a new field v, which differs from Ψ_{con} by the single-soliton phase factor,

$$v = \exp\left[-i\varphi - i\beta(t-y)\right]\Psi_{con}\,. \quad (1.6)$$

The field v can be written as a decomposition

$$\begin{pmatrix} v \\ v^* \end{pmatrix} = \int_{-\infty}^{+\infty} \frac{\mathrm{d}k}{2\pi}\left[a_k\varphi_k(x) + a_k^*\bar{\varphi}_k(x)\right]\,, \quad (1.7)$$

which is an analog of expansion over plane waves in homogeneous case. Here φ, $\bar{\varphi}$ are eigen-functions

$$\hat{L}_\eta\varphi_k = (k^2+\eta^2)\varphi_k, \qquad \hat{L}_\eta\bar{\varphi}_k = -(k^2+\eta^2)\bar{\varphi}_k, \quad (1.8)$$

of the operator

$$\hat{L}_\eta \equiv (\partial_t^2 - \eta^2)\hat{\sigma}_3 + \frac{2\eta^2}{\cosh^2[\eta(t-y)]}(2\hat{\sigma}_3 + i\hat{\sigma}_2)\,, \quad (1.9)$$

describing evolution of linear perturbation about the single soliton profile (1.4) of the no-disorder NLSE. They can be written as $\varphi_k = f_{k/\eta}(x)$ and $\bar{\varphi}_k = \bar{f}_{k/\eta}(x)$, where $x = \eta(t-y)$, and f_k, $\bar{f}_k$ are the eigen-functions of $\hat{L}_\eta$ at $\eta = 1$, defined in Appendix A. This complete system of the eigen-functions was found by Kaup in [11]. The coefficients a_k and a_k^* in Eq. (1.8) are functions of z. Besides, the eigen-functions φ_k and $\bar{\varphi}_k$ depend on z via $\eta(z)$ and $y(z)$, entering their definition. The functions φ_k, $\bar{\varphi}_k$ are orthogonal to the four localized modes, corresponding to variations of the four soliton parameters in (1.4) (see Appendix A), and the orthogonality conditions can be written as

$$\int_{-\infty}^{+\infty}\mathrm{d}t\,\cosh^{-1}(x)(v+v^*) = 0\,,$$
$$\int_{-\infty}^{+\infty}\mathrm{d}t\,\tanh(x)\cosh^{-1}(x)(v-v^*) = 0\,,$$
$$\int_{-\infty}^{+\infty}\mathrm{d}t\,x\cosh^{-1}(x)(v+v^*) = 0\,,$$
$$\int_{-\infty}^{+\infty}\mathrm{d}t\,[x\tanh(x)-1]\cosh^{-1}(x)(v-v^*) = 0\,. \quad (1.10)$$

The relations (1.10) fix uniquely (even though inexplicitly) the soliton parameters, introduced by Eq. (1.4), for a given function $\Psi(z,t)$.

Substitution of Eqs. (1.3-1.6) into the noisy NLSE (1.1) (where d has to be replaced by $1+\xi$) with subsequent expansion over ξ and v results in

$$\begin{aligned} &i\eta\partial_z\alpha f_0(x) - \partial_z\eta f_3(x) + \eta^2(\partial_z y - 2\beta)f_1(x) \\ &+ i\eta\partial_z\beta f_2(x) + \partial_z\begin{pmatrix} v \\ v^* \end{pmatrix} - i\hat{L}_\eta\begin{pmatrix} v \\ v^* \end{pmatrix} + \dots \\ &= i\xi\eta^3\left[\frac{1}{\cosh x} - \frac{2}{\cosh^3 x}\right]\begin{pmatrix} 1 \\ -1 \end{pmatrix}, \end{aligned} \quad (1.11)$$

where $x = \eta(t-y)$. Dots in (1.11) stand for high-order terms in v and β. Then, the equations for the soliton parameters and the continuous spectrum amplitudes a_k can be found by projecting the equation (1.11) onto respective eigen-functions of $\hat{L}_\eta$ (1.9). Let us present here an expansion of the right-hand side of (1.11) into a series over the eigen-functions

$$\begin{aligned} &i\left[\frac{1}{\cosh x} - \frac{2}{\cosh^3 x}\right]\begin{pmatrix} 1 \\ -1 \end{pmatrix} \\ &= \eta^{-1}\int\frac{\mathrm{d}k}{2\pi}(b_{k/\eta}\varphi_k + b^*_{k/\eta}\bar{\varphi}_k) - if_0(x)\,, \end{aligned} \quad (1.12)$$

$$b_q = \frac{\pi i}{2}\,\frac{(q+i)^2}{\cosh(\pi q/2)}\,. \quad (1.13)$$

required for further calculations. To derive Eq. (1.12), we use the relations (A11,A12) from Appendix A.

In the multi-soliton case a localized part of Ψ, Ψ_{sol}, can be approximated as an N-soliton solution of no-disorder NLSE with $4N$ parameters, varying in z. If individual solitons in the N-soliton pattern are well-separated from each other (i.e. if inter-soliton separations are all much larger than a single soliton width), Ψ_{sol} can be approximated (with an exponential accuracy over the inter-soliton separations) by a sum of single-soliton contributions

$$\begin{aligned} \Psi_{sol} = &\sum_{i=1}^{N}\frac{\eta_i}{\cosh\left[\eta_i(t-y_i)\right]} \\ &\times\exp\left[i\alpha_i + i\int_0^z\mathrm{d}z'\,\eta_i^2 + i\beta_i(t-y_i)\right]\,, \end{aligned} \quad (1.14)$$

where η_i, y_i, α_i and β_i are real parameters, standing for amplitudes, positions, phases and phase velocities of the solitons. Each soliton is labelled by its number i, $i = 1,\cdots,N$. Very much like in the single-soliton case, the soliton parameters, driven by the z-dependent disorder $\xi(z)$, depend on z. The $4N$ parameters of Ψ_{sol} are determined for a given Ψ through $4N$ conditions generalizing the relations (1.10). The conditions manifest orthogonality between the continuous spectrum and localized modes of differential operator defined for linear perturbation of the no-disorder version of (1.1) about its N-soliton solution.

We assume that a sequence of identical (of the same unit amplitude and zero initial phase velocity) ideal solitons are launched into the fiber at $z=0$. Thus the initial conditions for Ψ are

$$\eta_i(0) = 1, \quad \beta_i(0) = 0, \quad \Psi_{con}(0,t) = 0\,. \quad (1.15)$$

The initial positions of the solitons, $y_i(0)$, are parameters coding the transmitted information. Solitons phases, $\alpha_i(0)$, have to be included in the initial set up, also.

D. Weakness of disorder

The separation (1.3) of the entire solution of Eq. (1.1) into the localized and de-localized parts is natural in the case of weak disorder. The weakness of disorder ($D \ll 1$) has two important consequences: first, the radiation shed by soliton is also weak, i.e. $\Psi_{con} = O(\xi)$, and second, parameters of soliton vary slowly in z, while dynamics of the radiation field Ψ_{con} is relatively fast. The weakness of the radiation intensity, $|\Psi_{con}| \ll 1$, suggests linear description for Ψ_{con}. Let us, however, stress, that, generally, the decomposition (1.3), determined by Eqs. (1.4-1.9) for a single soliton (and by analogous relations for the multi-soliton case), does not require any smallness of Ψ_{con}. The generality of the approach will help us to construct a consistent perturbation theory (which, as we demonstrate below, requires an account for some higher order terms).

An important part of our further analysis will be focused on derivation and solution of a linear (as the radiation is weak) equation for Ψ_{con}. The equation gets a rather complex structure, which, generally, requires an accurate, case specific, analysis. However, the asymptotic behavior of Ψ_{con}, away from all the solitons, is simple and general, and it is certainly worth discussing it here, already in the introductory part. Far from solitons the radiation field Ψ_{con} is described by the linear wave equation

$$-i\partial_z \Psi_{con} = \partial_t^2 \Psi_{con} \,. \tag{1.16}$$

Thus, in the asymptotic domain the field, Ψ_{con}, can be expanded over the set of plain waves, $\propto \exp(-ik^2 z + ikt)$. We see, that at a given z the quantity k determines the frequency of the envelop. However, since t is the retarded time, then k has also a sense of the wave vector, the name we adopt in subsequent analysis. In the reference frame, moving with the light speed along the fiber, a wave packet with the wave vector k propagates along the z-axis with the group velocity $2k$. Therefore, the group velocity decays as the wavelength, k^{-1}, increases. This means, in particular, that short waves arrive first to remote points.

II. SHEDDING OF RADIATION BY A SINGLE SOLITON

The symmetry of the single-soliton set up allows reduction in number of essential degrees of freedom. Since both the equation (1.1) and the single-soliton version of the initial condition (1.15) are invariant under time inversion, $t \to -t$, neither soliton position, y, nor its phase velocity, β, are changing with z. The integral quantity $E = \int \mathrm{d}t\, |\Psi|^2$ (which is also natural to call energy, since it corresponds to the energy of original electromagnetic field) is conserved. This conservation law is due to the gauge symmetry of Eq. (1.1). The single-soliton version of the conservation law is

$$2\eta + \int \mathrm{d}t\, |v|^2 = 2 \,. \tag{2.1}$$

It gives an instantaneous (valid at any given z) relation between soliton amplitude and integral over t of the radiation intensity. The soliton phase, α, although changing under the action of disorder, does not enter (2.1). Notice, that the relation (2.1) is valid generally, regardless of the relative strength of the two terms on the left-hand side of Eq. (2.1).

The weakness of disorder ($D \ll 1$) is essential for the next two steps, which are:
1) *Linear* approximation, reducing calculations to direct account for the leading order in the radiation, ξ, terms in the basic dynamical equation. We will show below that the direct perturbation expansion is valid at $z \ll 1/D$, where deviations of η from unity are small;
2) *Quasi-Linear* approximation, explaining generalization of the pure linear approximation to the case of moderate- ($z \sim 1/D$) and long- ($z \gg 1/D$) haul transmissions. For such z a cumulative change of the soliton amplitude, η, becomes essential, while the radiation shed is still (like in the linear case) weak at any given position.

Equations for z-dependence of the parameters η, β, α, y, a_k and a_k^* are presented and discussed below separately for the linear ($z \ll 1/D$) and quasi-linear ($z \gtrsim 1/D$) cases. An essential part of the subsequent analysis (especially complicated in the quasi-nonlinear case) will be devoted to the proof of the following asymptotic statement: the higher-order terms (dots in (1.11)) do not contribute to the leading asymptotic description of the radiation profile v at any $t, z \gg 1$. Notice, however, that some of the higher-order terms have to be taken into account in the asymptotic equations for the soliton parameters.

A. Linear approximation

The linear (first order in ξ) approximation is examined in the Subsection. Recalling that the parameters α, β and η (and, also, y, if the soliton is not moving) are z-independent in the no-disorder ($\xi = 0$) case, one finds that z-derivatives of the slow variables are $O(\xi)$ or smaller. The radiation, v, is also $O(\xi)$, that is small due to the smallness of ξ. According to the conservation law (2.1), $\eta = 1 + O(v^2)$, i.e. it can be simply replaced by unity in the approximation. All the observations make it really simple to linearize (1.11) with respect to ξ.

Once the linearized equation is found, one can derive equations for the soliton parameters and the expansion coefficients a_k, introduced by Eq. (1.7), projecting the equation to the respective eigen-functions of the operator $\hat{L}$ (see Appendix A). Projection to the eigen-functions of the discrete spectrum gives the following equations for the soliton parameters

$$\partial_z \alpha = -\xi \,, \quad \partial_z \eta = 0 \,, \quad \partial_z \beta = 0 \,, \quad \partial_z y = 2\beta \,, \tag{2.2}$$

where we used the expansion (1.12). In agreement with what was already discussed, Eq. (2.2) shows that neither y nor β depend on z. Below we put $\beta = 0$ in accordance with the initial conditions, and assume $y = 0$ (without any loss of generality). Eq. (2.2) confirms an already mentioned observation that η does not get any z-dependence in the first order in ξ. Then, the equation for the continuous spectrum coefficients of the radiation expansion, a_k, derived from Eqs. (1.11,1.12,1.13), is

$$\partial_z a_k - i(k^2+1)a_k = b_k\xi\,, \tag{2.3}$$

where b_k is defined in Eq. (1.13). The solution of Eq. (2.3) is written as

$$a_k(z) = \int_0^z \mathrm{d}z'\xi(z')b_k \exp\left[i(k^2+1)(z-z')\right]\,. \tag{2.4}$$

Substituting Eq. (2.4) into Eq. (1.7) and considering the radiation far away from the soliton (that implies $t \gg 1$) one gets

$$v \approx -\frac{i}{4}\int_0^z \mathrm{d}z'\,\xi(z')\exp\left[-i(z-z')\right]\mathcal{J}\left(t, z-z'\right)\,, \tag{2.5}$$

$$\mathcal{J}(t,s) = \int \mathrm{d}q\,\frac{(q-i)^2}{\cosh[\pi q/2]}\exp(-iqt - iq^2 s)\,. \tag{2.6}$$

A stationary phase calculation of the integral on the right-hand side of Eq. (2.6) gives

$$\mathcal{J}(t,s)\approx\sqrt{\frac{\pi}{is}}\left(\frac{t}{2s}+i\right)^2\exp\left(i\frac{t^2}{4s}\right)\cosh^{-1}\left(\frac{\pi t}{4s}\right)\,. \tag{2.7}$$

The asymptotic expression (2.7) is valid at $s \gg 1$.

To describe the space-time dependence of the radiation, we examine the radiation intensity $|v|^2$, averaged over realizations of the disorder ξ, in the asymptotic domain of large z and t, $z, t \gg 1$. Multiplying two replicas of (2.5) to each other, and averaging the result over disorder, in accordance with Eq. (1.2), one finds

$$\langle|v|^2\rangle = \frac{D}{16}\int_0^z \mathrm{d}z'\,|\mathcal{J}(t,z-z')|^2\,. \tag{2.8}$$

At $t \gg 1$ one can substitute $\mathcal{J}$ in Eq. (2.8) by its asymptotics (2.7).

Let us first consider relatively short t, $t \ll z$. For z' in (2.8), restricted by $z-z' \gg t$, one gets $|\mathcal{J}|^2 \approx \pi/(z-z')$, resulting in the logarithmic divergence of the integral in (2.8) at small values of $z-z'$. The divergence is cut at $z-z' \sim t$, leading to the following radiation intensity profile

$$t \ll z \ll 1/D, \quad \langle|v|^2\rangle \approx \frac{\pi}{16}D\ln\frac{z}{t}\,. \tag{2.9}$$

In the domain of the radiation forerunner defined by, $t \gg z$, cosh in Eq. (2.7) can be replaced by its exponential asymptotics. Then, the integral in (2.8) is formed in the region of the shortest z' allowed in the domain. Calculating the integral explicitly, one derives the following asymptotics for the radiation forerunner

$$z \ll 1/D, \quad z \ll t, \quad \langle|v|^2\rangle \approx \frac{Dt^3}{32z^3}\exp\left[-\frac{\pi t}{2z}\right]\,. \tag{2.10}$$

The two asymptotic expressions (2.9) and (2.10) match at $z \sim t$.

It is instructive to present a qualitative explanation for the logarithmic profile (2.9). At small k the source of the radiation (localized at the soliton) can be treated as a point-like one. Therefore waves with the wave vectors $k < 1$ are excited by the disorder with approximately equal probability. Nevertheless, they have different group velocities. Among all the waves shed by the soliton (at $t \sim 1$ and $z' < z$) only those special with the wave vector (group velocity) $k \geq t/z$ contribute to $\langle|v(t)|^2\rangle$ at given z and t. On the other hand, emission of the waves with $k > 1$ is supressed. Thus, the main contribution to $\langle|v(t)|^2\rangle$ is proportional to $\int_{t/z}^1 dk/k = \ln(z/t)$, where the $1/k$ factor originates from the group velocity.

We conclude the Subsection establishing the region of validity for the linear approximation explained above. The first, and immediate, consequence of the linear approximation was the smallness of the soliton amplitude change. It means that the amount of energy shed by the soliton into radiation is negligible in comparison with the energy still left in the soliton, $E_{sol} \approx 2$. According to (2.9,2.10), the average energy shed into the radiation is, $E_{rad} = \langle\int \mathrm{d}t\,|v|^2\rangle$. One finds, that the radiation energy is mainly stored in the region, separating the logarithmic and the exponential profiles, i.e. $E_{rad} \sim Dz$. Since, according to (2.1), the overall energy is conserved, one finds that the linear approximation is justified, i.e. $E_{sol} \gg E_{rad}$, if z is essentially shorter than the degradation scale, $z_{degr} = 1/D$.

B. Quasi-Linear approximation.

Let us first draw a qualitative picture of what is happening at the scales larger than the degradation one, $z > z_{degr}$. Once z exceeds $z_{degr} = 1/D$, the balance of energy between the soliton and the radiation shifts towards the radiation. However, the differential (per unit z) release of energy into the radiation remains small and, actually, continues to decrease with z. The radiation emitted by soliton moves out of the soliton with a speed, fixed by the instantaneous value of the soliton amplitude η at the moment of emission, z. Once emitted the radiation never returns back to the soliton, i.e. it does not affect η later (at larger z). Therefore, if the density of radiation was small at the relatively short z, $z \ll 1/D$, (the fact proven in the previous Subsection) it cannot increase at the larger z, quite opposite, it may only decrease, i.e. $|v| \ll 1$ at any t and z. This feature of the linear approximation will be, therefore, carried over to the larger z. The only new ingredient (not considered at shorter z) is account for slow, but still a, degradation of the soliton with z. Physically, the quasi-linear approximation works because the waves shed by soliton leave it fast, while soliton passes the distance $\delta z \sim 1/\eta^2$, and the soliton amplitude η does not get any essential change during δz (since $D \ll 1$).

Our task here is, assuming some given dependence of η on z, to study the radiation profile v. Then, one derives from Eqs. (1.11,1.12,1.13)

$$\partial_z a_k - i(k^2+\eta^2)a_k = \eta^2 b_{k/\eta}\xi\,. \tag{2.11}$$

Some terms, originating from a z-dependence of η, were omitted in Eq. (2.11). This step will be justified below. The solution of Eq. (2.11) is

$$a_k(z) = \int_0^z \mathrm{d}z'\xi(z')\eta^2(z')b_{k/\eta(z')} \times \exp\left[ik^2(z-z') + i\int_{z'}^z \mathrm{d}z''\eta^2(z'')\right]. \tag{2.12}$$

Substituting Eq. (2.12) into Eq. (1.7) and considering the radiation away from the soliton (that implies $\eta t \gg 1$) one gets

$$v \approx -\frac{i}{4}\int_0^z \mathrm{d}z'\,\xi(z')\eta^3(z')\exp\left[-i\int_{z'}^z \mathrm{d}\zeta\,\eta^2(\zeta)\right] \times \mathcal{J}\left[\eta(z')t, \eta^2(z')(z-z')\right], \tag{2.13}$$

where the function $\mathcal{J}$ is defined by Eq. (2.6).

Eq. (2.13) is fundamental for further calculation of both the η dependence on z, and the average radiation intensity profile dependence on t and z. (The following two Subsections will be devoted specifically to the two aforementioned subjects). However, it is very important to justify beforehand the validity of those few but crucial assumptions made in the course of the Eq. (2.13) derivation from Eq. (1.11). The rest part of the present Subsection is devoted to the task.

The key question here is: if some small terms in (1.12), neglected in the course of derivation of (2.13), could be accumulated at the largest z? The major result will be negative answer to the question. To prove the general validity of Eq. (2.13) one divides the entire t-domain into two distinct regions, first, of some τ-wide soliton vicinity, $\tau \gg 1/\eta$, and the rest (remote region of t). The two regions will be considered separately. First, the validity of (2.13) should be proved for t from the box $[-\tau,\tau]$. Then, on the second step, one should take into account a term, omitted in the derivation of (2.13), originating from a z-dependence (via η) of the eigen functions φ_k and $\bar{\varphi}_k$, in Eq. (1.7).

The generalized version of Eq. (2.11), accounting for the dangerous term, is

$$\partial_z a_k - i(k^2+\eta^2)a_k + \hat{A}a_k = \eta^2 b_{k/\eta}\xi\,,$$

where $\hat{A}$ is a non-local over k and non-singular linear operator, estimated by, $\hat{A} \sim \partial_z\eta/\eta$. Assuming that the $\hat{A}$-correction is small, one arrives at the following modification of Eq. (2.13)

$$v \approx -\frac{i}{4}\int_0^z \mathrm{d}z'\,\xi(z')\eta^3(z')\exp\left[-i\int_{z'}^z \mathrm{d}z''\eta^2(z'')\right] \times \left[1 + \int_{z'}^z \mathrm{d}z''\hat{A}\right]\mathcal{J}\left[\eta(z')t, \eta^2(z')(z-z')\right]. \tag{2.14}$$

For t, bounded by the τ-wide box, integration over z' from the right-hand side of Eq. (2.13) is formed at $z - z' \sim \tau/\eta$. Therefore, correction to the integrand of Eq. (2.13) due to the $\hat{A}$-term in (2.14) is estimated by

$$(z-z')\hat{A} \sim \frac{\tau}{\eta^2}\partial_z\eta \sim D\tau\eta^3\,, \tag{2.15}$$

where one substitutes the law (0.1), announced in Introduction and derived in the next Subsection. The correction (2.15) is small provided $\tau \ll D^{-1}\eta^{-3}$. The later inequality is obviously compatible with the only restriction we have imposed so far on the size of the box, $\tau \gg \eta^{-1}$.

Next, we discuss the region of remote t, $|t| > \tau$, where the soliton part of the solution Ψ is negligible, while Ψ_{con} satisfies the linear wave equation (1.16). One can find Ψ_{con} outside the box by solving Eq. (1.16) with proper boundary conditions, where $\Psi_{con}(\pm\tau)$ was determined on the previous step, and it is also assumed that the radiation only escapes the τ-box but never re-enters. Fortunately, the result of this procedure coincides with the expression (2.13). Indeed, it is straightforward to check that Ψ_{con} related to v via the phase factor change (1.6), satisfies the linear equation (1.16), if v is given by Eq. (2.13). It is also seen from Eq. (2.7), that v contains only waves leaving the τ-box. All this proves that there are no essential corrections to Eq. (2.13) originating from the domain of remote t either.

C. Degradation law for soliton amplitude.

The energy balance between the soliton and the radiation controls the law of the soliton amplitude decay with z. From the basic equation (1.1) one gets

$$\partial_z|\Psi|^2 = id(z)\partial_t(\Psi^*\partial_t\Psi - \Psi\partial_t\Psi^*)\,. \tag{2.16}$$

This equation describes dynamics of the energy density $|\Psi|^2$, and leads to the conservation law (2.1). Integrating Eq. (2.16) over the τ-wide box, introduced in the previous subsection, one obtains a relation between the amount of energy shed by soliton and the flux of energy coming through the boundaries of the box. We choose τ to be large enough, so that the t-integral of $|\Psi|^2$ gets the major contribution from the soliton itself, and is equal to 2η. The integral of the right-hand side of Eq. (2.16) is reduced to two boundary terms at $t = \pm\tau$. At the boundaries one can replace Ψ by Ψ_{con} and, also, replace ξ by zero. The result is

$$\partial_z\eta(z) = i\left[v^*(z,\tau)\partial_\tau v(z,\tau) - v(z,\tau)\partial_\tau v^*(z,\tau)\right] \tag{2.17}$$

We show below, that the dependence of η on z can be established from Eq. (2.17) with its right-hand side replaced by its average value (over the disorder statistics). Performing this averaging, in accordance with Eq. (1.2), one arrives at

$$\partial_z\eta = \frac{Di}{8}\int_0^z \mathrm{d}z'\,\eta^6(z')\mathcal{I}^*\partial_\tau\mathcal{I}\,, \tag{2.18}$$

where $\mathcal{I}(z,t)=\mathcal{J}[\eta(z')\tau,\eta^2(z')(z-z')]$, and the function $\mathcal{J}$ is defined by Eq. (2.6). In Eq. (2.18) the function can be approximated by its asymptotic expression (2.7), resulting in

$$\partial_z\eta=-\frac{\pi D}{8}\int_0^z \mathrm{d}z'\,\frac{\tau\eta^4(z')}{(z-z')^2}\frac{(\zeta^2+1)^2}{\cosh^2(\pi\zeta/2)}\,,$$

where $\zeta=\tau/[\eta(z')(z-z')]$. The integral over z' in the expression is formed at $z-z'\sim\tau/\eta$. The size of the box τ can be chosen to be much smaller than ηz (if $z\gg 1$). Then, for relevant z', $z-z'\ll z$, and $\eta(z')$ can be substituted by $\eta(z)$. Passing from z' to the integration variable ζ and extending the integration region over ζ down to 0 (this is possible since $\tau/z\ll 1$) one gets

$$\partial_z\eta=-\frac{\pi D}{8}\eta^5\int_0^\infty \mathrm{d}\zeta\,\frac{(\zeta^2+1)^2}{\cosh^2(\pi\zeta/2)}=-\frac{8D}{15}\eta^5\,. \quad (2.19)$$

Integration of the differential equation (2.19) gives the final result for the degradation law (0.1), announced in the Introduction.

The law of the soliton decay given by Eq. (0.1) is deterministic in spite of the randomness of the initial set up, described by Eq. (1.1). The remarkable fact is due to the self-averaged feature of η, and the rest part of the paragraph is devoted to the proof of the statement. Indeed, we demonstrate, that deviation of η (for a given realization of the disorder ξ) from its average is small.

To establish statistical properties of η, we turn to the auxiliary quantity

$$\mathcal{V}(z)\equiv i\left[v^*(z,\tau)\partial_\tau v(z,\tau)-v(z,\tau)\partial_\tau v^*(z,\tau)\right],\quad (2.20)$$

standing on the right-hand side of Eq. (2.17). The irreducible pair correlation function (cumulant) of $\mathcal{V}$

$$\mathcal{K}(z_1,z_2)=\langle\mathcal{V}(z_1)\mathcal{V}(z_2)\rangle-\langle\mathcal{V}(z_1)\rangle\langle\mathcal{V}(z_2)\rangle\,,\quad (2.21)$$

is presented, according to (1.2,2.13), as a double integral over $z'_{1,2}$. One examines Eq. (2.21) at large values of $z_1>z_2$, $\eta\tau$, $z_{1,2},\eta\tau\gg 1$, and also assumes that the following two inequalities, $z_1-z_2\gg\eta^{-2}\tau$ and $(z_1-z_2)\partial_z\eta\ll 1$, are valid. Then, using Eq. (2.7), one finds

$$|\mathcal{K}|<D^2\eta^5\frac{\tau^5}{(z_1-z_2)^2}\,,\quad (2.22)$$

where the phase $x^2/4\Xi$ in Eq. (2.7) was dropped. (An account for the phase would decrease the value of the right-hand side in Eq. (2.22) and turn the inequality into equality.) Integrating Eq. (2.22) over some z_0-wide vicinity of $z=z_1$, one derives

$$\int_{z-z_0}^z \mathrm{d}z'\,|\mathcal{K}(z,z')|<D^2\eta^9\tau\,,\quad (2.23)$$

where $z_0\gg\tau/\eta$. Evaluating the inequality (2.23) further, one gets

$$\left\langle\left[\int_{z-z_0}^z \mathrm{d}z'\,\mathcal{V}(z')\right]^2\right\rangle\left\langle\int_{z-z_0}^z \mathrm{d}z'\,\mathcal{V}(z')\right\rangle^{-2}-1<\frac{\tau}{\eta z_0}\ll 1\,.$$

The integral, $\Delta\eta\equiv\int_{z-z_0}^z \mathrm{d}z'\,\mathcal{V}(z')$, determines variations of $\eta(z')$ for z' from the interval bounded by $z-z_0$ and z. We established that fluctuations of $\Delta\eta$ are weak. On the other hand, we are free to choose such z_0 that $\Delta\eta\ll\eta$. To conclude, evolution of η can be described in terms of the deterministic equation (2.19).

D. Average Radiation

This subsection is devoted to derivation of the average radiation intensity profile from Eqs. (0.1,1.2,2.13). We examine it in the asymptotic domain of large z and t, $z,t\gg 1$. Averaging the radiation intensity $|v|^2$ in accordance with (1.2) one finds

$$\langle|v|^2\rangle=\frac{D}{16}\int_0^z \mathrm{d}z'\eta^6\left|\mathcal{J}[\eta t,\eta^2(z-z')]\right|^2,\quad (2.24)$$

where $\eta=\eta(z')$, and $\mathcal{J}$ is defined by Eq. (2.6).

The radiation profile at $z\gg 1/D$ gets a more complicated structure than in the domain of short z, $z\ll 1/D$, studied above in the Section II A. Using the asymptotic expression (2.7) for the auxiliary function $\mathcal{J}$ and substituting Eq. (0.1) into Eq. (2.24) one derives

$$\langle|v|^2\rangle=\frac{15\pi}{512}\int_0^z\frac{\mathrm{d}z'}{(z-z')[z'+(15/32)D^{-1}]}\times\left[\frac{t^2}{4\eta^2(z-z')^2}+1\right]^2\cosh^{-2}\left[\frac{\pi t}{4\eta(z-z')}\right].\quad (2.25)$$

Analysis of this expression shows that there are three different asymptotic domains of t for any given z:

(a)	$t\ll[z^3/D]^{1/4}$	&	$zD\gg 1$,
(b)	$[z^3/D]^{1/4}\ll t\ll z$	&	$zD\gg 1$,
(c)	$t\gg z$	&	$zD\gg 1$.

In the domain (a) two different asymptotic regions of z', $1/D\ll z'\ll z$ and $t/\eta\ll z-z'\ll z$, give the major contribution into the integral on the right-hand side of Eq. (2.25). Collecting the major logarithmic terms, one obtains

$$\text{(a)}\qquad\langle|v|^2\rangle=\frac{15\pi}{512z}\ln\frac{D^{3/4}z^{7/4}}{t}\,.\quad (2.26)$$

In the domain (b) the major contribution is coming from the $1\ll Dz'\ll(z/t)^4$ region of z' integration in (2.25), leading to

$$\text{(b)}\qquad\langle|v|^2\rangle=\frac{15\pi}{128z}\ln(z/t)\,.\quad (2.27)$$

And, finally, at $t\gg z$ the integral in (2.25) is formed at $Dz'\lesssim z/t$, where cosh can be substituted by its exponential asymptotics. This leads to

$$\text{(c)}\qquad\langle|v|^2\rangle=\frac{15t^3}{256z^4}\exp\left(-\frac{\pi t}{2z}\right).\quad (2.28)$$

Once all the asymptotics (for short z in the previous Subsection, and for long z here) are presented let us describe a general picture of the radiation distribution. The radiation front runs out of the soliton with the constant speed, $t/z \sim 1$. The logarithmic profile is formed behind the front, while the radiation forerunner decays exponentially with $t/z \gg 1$. Energy of the radiation is contained mainly in the boundary region between the logarithmic and the exponential profiles. At $z \ll D^{-1}$, the logarithmic profile (2.9) is simple, and the pre-exponential factor depends on D, as it seen from Eq. (2.10). At the larger z, $z \gg 1/D$, when the soliton has already shed almost all its energy into the radiation, the logarithmic profile splits into two parts described by Eq. (2.26) and Eq. (2.27), respectively, and the exponential asymptotics is modified to Eq. (2.28). The regime (a) is formed by the waves with $k < \eta$ emitted continuously at different z', whereas the regimes (b) and (c) are formed by the "fast" waves, emitted at z' far from the observation point z. The boundary between the regimes (a) and (b) is determined by the condition $t \sim \eta z$ (that is the "distance" passed by waves with $k \sim \eta$). The profile in the regime (a) knows about the current size η of the soliton, whereas in the regimes (b) and (c) the radiation is insensitive to the current value of η. Note, that the universal profile, formed in the regions (b) and (c), does not depend on the intensity of the disorder, D, and the only information stored in the asymptotics is about the initial soliton profile. The universal profile (b), (c) is self-similar: $\langle |v^2| \rangle = z^{-1}\Phi(t/z)$. From the first sight, the type of this self-similarity, $t \sim z$, contradicts to the asymptotic equation (1.16). The confusion has a simple resolution. The main dependence of v on t is associated with its phase, which, as it is seen from Eq. (2.7), has normal kind of self-similarity $z \sim t^2$. However, the phase drops from the absolute value $|v|^2$, so that the self-similarity of the later object is determined by the subleading, $\sim 1/z$, terms in the eiconal approximation. Notice, that the phase (normal) self-similarity will be seen in the, mediated by radiation, two soliton interaction, we are switching our attention to in the next Section.

III. INTERACTION OF TWO SOLITONS

Propagation of a two-soliton pattern at intermediate z, $1 \ll z \ll 1/D$, is discussed in the current Section. As it was shown in Section II, dynamics of a single pulse within the range of scales bounded from above by the degradation scale, $z_{degr} = 1/D$, is trivial: y and β do not evolve, while the change of the soliton amplitude η is $O(zD)$, i.e. negligible. The major observation following from our analysis here is that the intersoliton separation, $y_2 - y_1$, coupled to the phase velocities $\beta_{1,2}$ of solitons in a two-soliton pattern, does get a non-trivial dynamics at the scales much shorter than the single soliton degradation scale, z_{degr}. We show that the inter-soliton interaction mediated by disorder is essential at the shorter scales $\sim z_{int}$, that is $1 \ll z_{int} \ll z_{degr}$. The soliton parameters $\beta_{1,2}$ are $O(\Psi_{con}^2)$, while Ψ_{con} itself is $O(\xi)$. Therefore, we divide our analysis into the following steps. First, the radiation Ψ_{con} will be related to ξ in the linear approximation. Second, $\beta_{1,2}$, and then $y_{1,2}$, will be presented as a second order form in Ψ_{con}. Finally, we will calculate statistics of the forces acting on the solitons and, therefore, will explain jitter the solitons experience.

A. Radiation generated by two solitons.

We consider the $N = 2$ case of the general setting (1.3,1.14) with the two well-separated solitons, $y = y_2 - y_1 \gg 1$ ($y_2 > y_1$ is assumed). At $z \ll D^{-1}$ one can substitute $\eta_1 = \eta_2 = 1$, and the localized part of Ψ (1.14) is reduced to

$$\Psi_{sol} = \frac{e^{i\alpha_1 + iz + i\beta_1(t-y_1)}}{\cosh(t-y_1)} + \frac{e^{i\alpha_2 + iz + i\beta_2(t-y_2)}}{\cosh(t-y_2)}. \quad (3.1)$$

The delocalized part, Ψ_{con}, of the complete solution (1.3) of Eq. (1.1) is built according to the general scheme outlined in Section I.

Similarly to Eq. (1.6), one introduces an auxiliary radiation field, v, $v = \Psi_{con}\exp(-i\alpha_1 - iz)$, accounting for the phase shift of the soliton, positioned at y_1. The field v can be written in the form of the expansion (1.7) over the continuous spectrum eigen functions of an auxiliary perturbation problem. The auxiliary problem is fixed by the operator $\hat{\mathcal{L}}$, which is a two-soliton generalization of the single-soliton operator (1.9). With the exponential accuracy over the separation $y = y_2 - y_1$, the differential operator $\hat{\mathcal{L}}$ is $\hat{\mathcal{L}} = \hat{L}(t - y_1)$ at $t < (y_1 + y_2)/2$ and $\hat{\mathcal{L}} = \hat{L}_\alpha(t - y_2)$ at $t > (y_1 + y_2)/2$. Here $\alpha = \alpha_2 - \alpha_1$ is the phase mismatch, $\hat{L}$ and $\hat{L}_\alpha$ are defined in Appendix A by Eqs. (A3,A13). We adopt the same general notations, φ_k, $\bar{\varphi}_k$ for the continuous spectrum eigen-functions of $\hat{\mathcal{L}}$, i.e. $\hat{\mathcal{L}}\varphi_k = (k^2+1)\varphi_k$, $\hat{\mathcal{L}}\bar{\varphi}_k = -(k^2+1)\bar{\varphi}_k$. The eigen-functions are fixed by their asymptotic behavior at large negative t:

$$t \to -\infty \quad \varphi_k \to \left(\frac{k-i}{k+i}\right)^2 \exp(ikt - iky_1)\begin{pmatrix}0\\1\end{pmatrix}. \quad (3.2)$$

Then, with the exponential accuracy, $\varphi_k = f_k(t - y_1)$ if $t < y_2$ and

$$\varphi_k(t) = \frac{(k+i)^2}{(k-i)^2}\exp(iky + i\alpha) f_{\alpha,k}(t - y_2)$$

if $t > y_1$. Here $y = y_2 - y_1$ and the functions f_k, $f_{\alpha,k}$ are defined by the expressions (A6,A7,A14). In the transient region $1 \ll t - y_1$, $y_2 - t \gg 1$, the two asymptotics of φ_k, presented above, coincide. One should also add, $\bar{\varphi}_k = \hat{\sigma}_1\varphi_k^*$, to the set of eigen-functions to make it complete. The orthogonality properties of φ_k, $\bar{\varphi}_k$ are identical to the ones given by Eqs. (A11).

The linear equation for v follows from the direct expansion of the basic equation (1.1),

$$\partial_z \begin{pmatrix} v \\ v^* \end{pmatrix} - i\hat{\mathcal{L}} \begin{pmatrix} v \\ v^* \end{pmatrix} + \cdots = g\xi \tag{3.3}$$

$$g = i\left[\frac{2}{\cosh^3(t-y_1)} - \frac{1}{\cosh(t-y_1)}\right]\begin{pmatrix} 1 \\ -1 \end{pmatrix} + i\left[\frac{2}{\cosh^3(t-y_2)} - \frac{1}{\cosh(t-y_2)}\right]\begin{pmatrix} e^{i\alpha} \\ -e^{-i\alpha} \end{pmatrix} \tag{3.4}$$

Here dots stand for terms corresponding to the localized modes, and $\hat{\mathcal{L}}$ was introduced above. Substituting the decomposition (1.7) into Eq. (3.3) and expanding its right-hand side over the eigen-functions of the operator $\hat{\mathcal{L}}$, one gets

$$\partial_z a_k - i(k^2+1)a_k = B_k\xi\,, \tag{3.5}$$

$$B_k = b_k\left[1 + \frac{(k-i)^2}{(k+i)^2}e^{-iky-i\alpha}\right], \tag{3.6}$$

where b_k are defined by Eq. (1.13). (In the derivation we did not account for z-dependence of φ_k, since $\partial_z\varphi_k = O(\xi)$). A solution of the Eq. (3.5) is

$$a_k(z) = \int_0^z \mathrm{d}z'\,\xi(z')\,\exp\left[i(k^2+1)(z-z')\right]B_k\,, \tag{3.7}$$

analogously to Eq. (2.4).

In the linear approximation over ξ, the solitons parameters can be examined in the framework of the same Eqs. (3.3,3.4). The resulting equations for the soliton parameters are

$$\partial_z\alpha_{1,2} = -\xi, \quad \partial_z\beta_{1,2} = 0, \quad \partial_z y_{1,2} = 2\beta_{1,2}\,, \tag{3.8}$$

similarly to Eq. (2.2). Note, that according to Eqs. (3.8) $\partial_z(\alpha_2-\alpha_1)=0$, i.e. the phase mismatch $\alpha=\alpha_2-\alpha_1$ is independent of z in the approximation.

B. Evolution of soliton parameters

As follows from Eqs. (3.8), the soliton parameters $y_{1,2}$ and $\beta_{1,2}$ do not get any z-dependence in the first order in v. One expects that in the next, second order, the β-equations gets a nonzero contribution, i.e. $\partial_z\beta_{1,2}\sim|v|^2$. Then, according to the y-equations (the last ones in Eq. (3.8)) fluctuations of the separation $y=y_2-y_1$ are estimated by $z^2|v|^2$, and are not small in the interesting range of scales, $z\sim z_{int}\ll z_{degr}$. The estimations also show that higher order, $O(v^3)$, corrections to the equations for $\beta_{1,2}$ are not essential. Further, it is easy to check that the equations for β contain the phases $\alpha_{1,2}$ only in the combination, $\alpha=\alpha_2-\alpha_1$. According to the first equation in (3.8), α does not evolve in the first order in ξ, while account for the next (second) order correction to the equation for α is inessential in the considered range of z, $z\ll 1/D$. To conclude, the only thing left to be studied is the second order in v contributions to the equations for $\beta_{1,2}$.

To find the contribution, one expands the basic equation (1.1) up to the second order in v

$$\partial_z\begin{pmatrix}\Psi \\ \Psi^*\end{pmatrix} = \cdots + i\xi\partial_t^2\begin{pmatrix}\Psi_{con} \\ -\Psi^*_{con}\end{pmatrix} + 2i\begin{pmatrix} 2|\Psi_{con}|^2 & \Psi^2_{con} \\ -(\Psi^*_{con})^2 & -2|\Psi_{con}|^2\end{pmatrix}\begin{pmatrix}\Psi_{sol} \\ \Psi^*_{sol}\end{pmatrix}, \tag{3.9}$$

where dots stand for the first-order terms. Extracting terms, proportional to $\partial_z\beta_1$, $\partial_z y_1$ from the left-hand side of Eq. (3.9) and making the respective projections one arrives at

$$\partial_z\beta_1 = \mathcal{F}(z) = \mathcal{F}_{vv}(z) + \mathcal{F}_{\xi v}(z) + \mathcal{F}_{\xi\alpha}(z)\,, \tag{3.10}$$

$$\mathcal{F}_{vv} = \int \mathrm{d}x\,\frac{\tanh x}{\cosh^2 x}\left[4|v|^2 + v^2 + (v^*)^2\right], \tag{3.11}$$

$$\mathcal{F}_{\xi v} = \xi\mathrm{Re}\int \mathrm{d}x\,\frac{\tanh x}{\cosh x}\partial_x^2 v\,, \tag{3.12}$$

$$\mathcal{F}_{\xi\alpha} = -\partial_z\alpha_1\mathrm{Re}\int \mathrm{d}x\frac{\tanh x}{\cosh x}v\,, \tag{3.13}$$

$$\partial_z y_1 = 2\beta_1 + \mathcal{P}_1, \tag{3.14}$$

$$\mathcal{P}_1 = i\int\frac{\mathrm{d}x\,x}{\cosh^2 x}\left[v^2 - (v^*)^2\right], \tag{3.15}$$

where $x=t-y_1$. For completeness, we calculated the second-order term in the equation for y_1, which in Eq. (3.14) is added to the first-order one. Expressions for the soliton positioned at $t=y_2$, can be obtained in a similar way. Using mechanical analogy, one can call β momentum of the soliton. Then $\mathcal{F}$ is the force, acting on the soliton, and $\mathcal{P}_1$ is an additional impulse.

One is interested to describe fluctuations (statistics) of y_1 as a function of z, assuming that the inter-soliton separation, $y=y_2-y_1$ is much larger than unity, but much less than z. Integrating the equations (3.10,3.14), we obtain

$$\delta y_1 = \int_0^z \mathrm{d}z'(2\beta_1+\mathcal{P}_1), \quad \beta_1 = \int_0^z \mathrm{d}z'\mathcal{F}(z_2). \tag{3.16}$$

According to the Central Limit Theorem [23], $\beta_{1,2}$ and $y_{1,2}$, as z-integrals of random functions, are Gaussian random processes at large z. This Gaussianity allows to estimate fluctuations of various quantities (about respective average values) for particular realization of the disorder, say, $|\delta y_1|$ fluctuates about $\langle(\delta y_1)^2\rangle^{1/2}$ with the same amplitude $\langle(\delta y_1)^2\rangle^{1/2}$.

The main contribution to δy_1 is related to the force $\mathcal{F}$. As it is shown in Appendix B, the average value of $\mathcal{F}$ is negligible (more accurately it is exponentially small in y, $\sim\exp(-y)$ and vanishes algebraically with $z\to\infty$). The cancellation (lack of a $\sim D$ contribution into the average value of the force $\mathcal{F}$) is a consequence of the reflectiveless feature of the solitons radiation. Thus fluctuations of β_1 are controlled by the pair correlation function of $\mathcal{F}$, calculated in detail in Appendix B. The main contribution to the correlation function is

$$\langle \mathcal{F}(z)\mathcal{F}(z')\rangle = D^2 G\delta(z-z')\,, \tag{3.17}$$

where G is given by Eq. (B27), $G \approx 0.14$. One, therefore, obtains from Eqs. (3.16,3.17)

$$\langle \beta_1^2(z)\rangle = D^2 Gz, \quad \langle (\delta y_1)^2\rangle = \frac{4}{3} GD^2 z^3. \tag{3.18}$$

Thus the typical change of the soliton position (counted from its initial value at $z = 0$) is estimated as $\delta y_1 \sim Dz^{3/2}$. The soliton leaves the allocated slot (in the soliton pattern), i.e. δy becomes $O(1)$, at $z \sim z_{int}$, $z_{int} = D^{-2/3} \ll 1/D \ll 1$. This happens well before soliton amplitude acquires any significant reduction, therefore justifying our approximation.

Note, that the average of the impulse, $2\int \mathrm{d}z\,\mathcal{F} + \mathcal{P}_1$, is equal to $2D/3$ (see Appendix B). That implies a systematic drift $2Dz/3$ in y_1. This drift is negligible in comparison with the fluctuating part of y_1, $\delta y_1 \sim Dz^{3/2}$, at $z \gg 1$.

It is of interest also to examine the relative motion of the solitons. Then one should take into account that the forces, acting on the solitons, actually interfere. The cross correlation term of the forces is dependent on the solitons phase mismatch α. It results in the following expression for the fluctuations of the relative position $y = y_2 - y_1$ (see Appendix B for details of the derivation)

$$\langle (\delta y)^2\rangle = \frac{8[1+\cos(2\alpha)]}{3} D^2 Gz^3. \tag{3.19}$$

Substituting here the approximate value $G \approx 0.14$, one arrives at Eq. (0.2) from the Introduction. Eq. (3.19) shows that fluctuations of the solitons separation are sensitive to the phase mismatch (e.g. the fluctuations are strongly suppressed at $\alpha = \pi/2$).

IV. MULTISOLITON CASE

Let us discuss the effect of soliton interaction in a multi-soliton pattern. The reflectiveless feature of the radiation guarantees lack of the radiation screening. In other words, all solitons positioned on distances $\lesssim z$ from a given soliton are affected by the radiation shed by the given soliton. Therefore, the radiation v in a vicinity of a given soliton is given by a superposition of a single-soliton radiative contributions, which differ by shifted phases only from the two-soliton case. Each of the contributions is only weekly dependent on the inter-soliton separation, provided the separation between the solitons is less than z (then the analysis similar to one produced in the Appendix B is correct). To conclude, force acting on a single soliton should grow with N, which is the number of solitons affecting the given soliton by their radiation.

To obtain quantitative conclusions, one extends the analysis of Appendix B to the multi-soliton case. Average force, applied to a soliton vanishes (at large z and if the exponential in y corrections, $\sim \exp(-y)$, are not taken into account). Fluctuations of y_i, β_i are Gaussian again (due to the Central Limit Theorem). One finds that the pair correlation function of the force acting on a given soliton (and also the pair correlation function of the given soliton position shift) is $\propto N$. Notice also, that like in the two-soliton case, force acting on the soliton, and thus change in the soliton position, is sensitive to the relative phases of all the N-solitons. However, unlike in the two-soliton case, it is impossible to suppress fluctuations of all the inter-soliton separations adjusting the soliton phases.

One concludes, that in the multi-soliton case Eqs. (3.18) for the velocity and the soliton position change acquiring an extra factor N. If the information rate in a fiber is fixed, N grows linearly with z, i.e. δy is estimated by $\sim \sqrt{\mu} Dz^2$, where μ is the number of solitons per unit length of the fiber.

V. DIRECT NUMERICAL SIMULATION

We discuss here Direct Numerical Simulations of the one- and two- soliton patterns. The major numerical problem here is due to long haul (large z) nature of the transmission. The radiation moves away from the soliton pattern and eventually hits the boundaries of the computational domain, which, in reality, cannot be infinite. Therefore, it is extremely important to design a method by which the radiation does not retracts from the boundaries, but instead evolves like it does not feel the artificial boundaries. The problem of numerically absorbing boundary conditions design is one of the typical computational problems in the wave-type equations, and numerous efforts have been made to overcome these numerical artifacts [25–27]. A common approach, widely used to overcome the numerical problem, is to apply an artificial damping at the vicinity of edges to suppress the radiation in the far region. However, during the evolution of the soliton, the transmission and reflection of waves takes place simultaneously. In other words, damping, inevitably creates a parasite back refraction of waves.

We solve the problem in other way. Namely, we introduce boundary conditions so that reflectionless feature of the artificial boundaries is controlled analytically. The only, but crucial, assumptions of the approach is that the intensity of the signal at the boundaries of the computational domain is low enough, so that one can linearize the basic Eq. (1.1) there. Let us consider regions $|t| \gg 1$ where one should observe the radiation going away from the solitons. In the region one can use the equation

$$(i\partial_z + \partial_t^2)\Psi = 0\,, \tag{5.1}$$

which is just the linear Schrödinger equation (without potential). The radiative boundary conditions, imposed on a solution of the equation (5.1) at the boundaries of the computational domain, $t = \pm T$, can be written as

$$-i\partial_t \Psi(z,T) = \sqrt{i\partial_z}\,\Psi(z,T)\,,$$
$$-i\partial_t \Psi(z,-T) = -\sqrt{i\partial_z}\,\Psi(z,-T)\,, \qquad (5.2)$$

where $\sqrt{i\partial_z}$ is a non-local (integral) operator:

$$\sqrt{i\partial_z}\,\Psi \equiv \sqrt{\frac{i}{\pi}}\int^z \frac{\mathrm{d}z_1}{\sqrt{z-z_1}}\partial_z\Psi(z_1)\,.$$

(The condition, $T \gg 1$, should also be satisfied.) Notice, that similar scheme for transient linear Schrödinger equation with a potential bounded in a finite domain was suggested in [27]. Furthermore, for the one-dimensional NLSE, the transparent boundary conditions have been discussed and introduced in several articles from various application fields (see e.g. [28, 29]).

Implementing this transparent boundary condition to a symplectic scheme for NLSE, we examined, first, degradation of single soliton, and then, interaction of two solitons caused by fluctuations of the dispersion coefficients. We use a standard random number generator to produce Gaussian zero-mean random process correlated at z_{var} with amplitude d_{var}. Choosing small z_{var} (z_{var} is 0.05 in our numerical experiments) we guarantee that the numerical random process approximates well the zero mean δ-correlated uniform noise for ξ described by $\langle\xi(z_1)\xi(z_2)\rangle = D\delta(z_1 - z_2)$, with $D = d_{var}^2 z_{var}$.

The results of the numerical simulations can be plotted in graphs. Fig. 1 shows dependence of the soliton amplitude on z, with the strength of disorder D equal to 0.0225. (D is chosen to be a small number to allow a quantitative comparison with the asymptotic theory, valid at $D \ll 1$.) Solid and dashed curves represent theory, resulted in Eq. (0.1), and DNS for a representative realization of the disorder, respectively. A good quantitative agreement between the theory and numerics is reported within an extremely long range of z.

We also perform direct numerical simulations of the two-soliton interaction. Notice that the two-soliton case requires using more accurate numerical definition of the soliton position at any given z. Since the soliton amplitude only weakly deviates from unity, the position of a soliton was found simply by minimization of $\sum_i[|\Psi(t_i,z)| - 1/\cosh(t_i - y)]^2$, where i numbers the temporal grid points in a vicinity of a special point where $|\Psi(t_i,z)|$ reaches its maximum. Fig. 2 shows dependence of the dispersion in the inter-soliton separation fluctuations, $\langle(\delta y)^2\rangle$, on z at the phase mismatches $\alpha = 0, \pi/4, \pi/2$. Our averaging (for each α) is done over 15 realizations. Numerical curves are solid, dashed curves correspond to theoretical predictions of Eq. (0.2). The strength of the disorder is chosen to be much smaller here than in the DNS for the single soliton decay, $D = 0.0125^2$, in purpose. We aimed to separate $r_{degr} = 1/D$ and the interaction scale, $r_{int} \sim D^{-2/3}$, as much as we can to be able to study the inter-soliton dynamics of the solitons with yet bare (non-perturbed) shape ($\eta = 1$) at $z \sim z_{int}$. The initial separation $y(0)$ was chosen to be large enough ($y(0) = 20$ for the data shown on Fig. 2) to avoid interference of the effects driven by disorder in dispersion coefficient with the direct interaction of solitons (the direct effect decays exponentially with the separation y [30, 31]). The Figure shows a good agreement between theory and numerics. To illustrate realization-dependence of δy we show it in Fig. 3 as a function of z for the 15 realizations at $\alpha = 0$. For comparison, the mean-square displacement is also shown in the Figure.

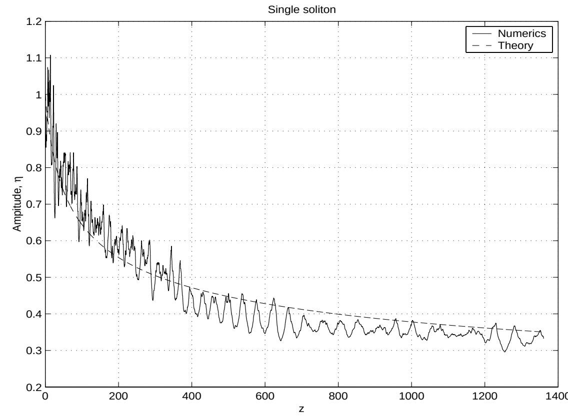

FIG. 1: Dependence of the soliton amplitude on z, position along the fiber, for disorder of the strength $D = 0.0225$. Solid and dashed curves represent theory, resulted in Eq. (0.1), and DNS for a representative realization of the disorder, respectively.

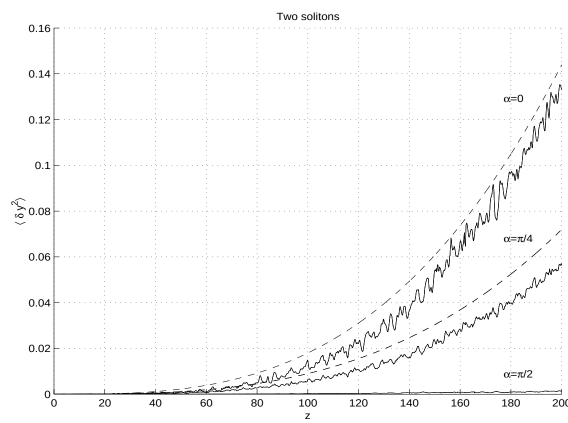

FIG. 2: The dependence of the dispersion of the inter-soliton separation fluctuation, $\langle(\delta y)^2\rangle$, on z, position along the fiber. The strength of disorder is $D = 0.0125^2$. Three different set of curves for the three different values of the inter-soliton phase mismatch, $\alpha = 0, \pi/4, \pi/2$, are shown. Dashed curves represent analytical result given by Eq. (0.2). Solid curves represent results of DNS, each averaged over 15 different realizations of disorder.

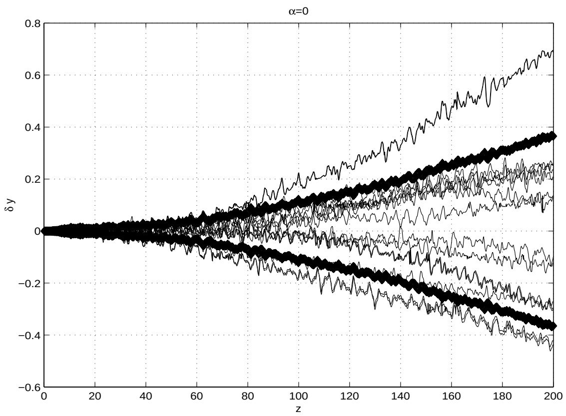

FIG. 3: Dependence of δy on z for 15 realizations is shown. The bold curves correspond to $\pm\sqrt{\langle(\delta y)^2\rangle}$ where average is taken over the realizations.

VI. CONCLUSION AND DISCUSSION

Let us recall different stages and scales in evolution of soliton patterns, which appear due to weakness of disorder, $D \ll 1$. The distance passed by soliton during one full turnover of its phase is unity in our notations. Soliton starts to degrade, i.e. its amplitude change becomes of the order of the initial value, at $z_{degr} = 1/D$. However, an interesting physics is also taking place at much shorter z. The inter-soliton interaction caused by radiation leads to an essential shift of the solitons at $z \sim z_{int}$, $z_{int} = N^{-1/3}D^{-2/3}$, where N is the number of solitons in the channel.

The major effect reported in the paper is the emergence of the separation-independent, fluctuating in z interaction between solitons, mediated by their mutual radiation. A frozen (t-independent), disorder (which produces a multiplicative noise in the NLSE) stimulates the shedding of radiation by solitons, which, in turn, mediates the inter-soliton interaction. The interaction causes the soliton to jitter randomly. The soliton displacement δy is zero mean Gaussian random variable, with the typical value estimated by $\delta y \sim Dz^{3/2}N^{1/2}$. If N does not grow with z (e.g. there are only finite number of solitons propagating in the channel) the z-dependence of the jitter is the same as the one given by the Elgin-Gordon-Haus jitter [32–35] developed under the action of random additive noise (short-correlated both in t and z noise of amplifiers in the fiber system). However, if the flow of information is continuous, i.e. if the front of radiation shed by the given soliton sweeps more and more solitons with z increase, $N \propto z$, the efficiency of the interaction grows with z in a faster, $\delta y \propto z^2$, pace, thus overscreening the Elgin-Gordon-Haus jitter in the long-haul transmission.

The algebraic in separation, i.e. long range, character of the inter-soliton interaction discussed in the manuscript is related to the reflectiveless feature of the radiation scattering on soliton(s). Notice, however, that the scattering becomes reflective in some nonintegrable generalizations of the NLS equation that are of physical importance, e.g. if random birefringence of fiber (Polarization Mode Dispersion) is taken into account [36]. The reflectivity leads to essential changes in the properties of the radiation and the inter-soliton interaction, e.g. force exerted on a soliton acquires nonzero mean.

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APPENDIX A: KAUP PERTURBATION TECHNIQUE

Recall some, known after the works [11, 12], properties of the perturbations near an ideal soliton described by the nonlinear Schrödinger equation

$$-i\partial_z\Psi = \partial_t^2\Psi + 2|\Psi|^2\Psi \,. \tag{A1}$$

Substituting the expression

$$\Psi = \left[\cosh^{-1}(t) + v\,\right]\exp(iz + i\alpha)\,,$$

into Eq. (A1) and expanding the result over v one finds

$$i\partial_z\begin{pmatrix} v \\ v^* \end{pmatrix} + \hat{L}\begin{pmatrix} v \\ v^* \end{pmatrix} = 0 \tag{A2}$$

where the operator $\hat{L}$ is

$$\hat{L} = \left(\partial_t^2 - 1\right)\hat{\sigma}_3 + \frac{2}{\cosh^2[t]}\left(2\,\hat{\sigma}_3 + i\,\hat{\sigma}_2\right), \tag{A3}$$

and the standard notations for the Pauli matrixes, $\hat{\sigma}_{1,2,3}$, are used. $\hat{L}$ satisfies the following set of relations

$$\hat{\sigma}_1\hat{L}\hat{\sigma}_1 = -\hat{L}^*\,, \qquad \hat{L}^+ = \hat{\sigma}_3\hat{L}\hat{\sigma}_3\,. \tag{A4}$$

Eigen set of the operator $\hat{L}$ solves

$$\hat{L}\,f = \lambda\,f, \tag{A5}$$

where f is an eigen-function correspondent to the eigenvalue λ. A general solution of (A5) is

$$f_k = \exp[i\,k\,t]\left\{1 - \frac{2\,i\,k\,\exp(-t)}{(k+i)^2\cosh(t)}\right\}\begin{pmatrix}0\\1\end{pmatrix} + \frac{\exp(i\,k\,t)}{(k+i)^2\cosh^2(t)}\begin{pmatrix}1\\1\end{pmatrix}, \quad \lambda_k = k^2 + 1\,, \tag{A6}$$

where k runs from $-\infty$ to $+\infty$. According to Eq. (A4), $\bar{f}_k \equiv \hat{\sigma}_1 f_k^*$ are other eigen functions of $\hat{L}$

$$\bar{f}_k = \exp(-i\,k\,t)\left\{1 + \frac{2\,i\,k\,\exp(-t)}{(k-i)^2\cosh[t]}\right\}\begin{pmatrix}1\\0\end{pmatrix} + \frac{\exp(-i\,k\,t)}{(k-i)^2\cosh^2(t)}\begin{pmatrix}1\\1\end{pmatrix}, \quad \lambda_k = -(k^2+1). \tag{A7}$$

The eigen set of $\hat{L}$ also contains the following marginally stable modes:

$$f_0 = \frac{1}{\cosh(t)}\begin{pmatrix}1\\-1\end{pmatrix}, \quad f_1 = \begin{pmatrix}1\\1\end{pmatrix}\frac{\tanh(t)}{\cosh(t)}, \tag{A8}$$

where $\lambda_0 = \lambda_1 = 0$. The existence of double poles at $k = \pm i$ means that two more functions must be added to the eigen-set for completeness

$$f_2 = \frac{t}{\cosh(t)}\begin{pmatrix}1\\-1\end{pmatrix}, \quad \hat{L}\,f_2 = -2f_1\,; \tag{A9}$$

$$f_3 = \frac{t\tanh(t)-1}{\cosh(t)}\begin{pmatrix}1\\1\end{pmatrix}, \quad \hat{L}\,f_3 = -2f_0\,. \tag{A10}$$

Next, $f_k^+\hat{\sigma}_3$ and $\bar{f}_k^+\hat{\sigma}_3$ are the left eigen-functions of $\hat{L}$, which satisfy

$$\int\limits_{-\infty}^{+\infty} \mathrm{d}t\,\bar{f}_k^+\hat{\sigma}_3\bar{f}_q = -\int\limits_{-\infty}^{+\infty} \mathrm{d}t\,f_k^+\hat{\sigma}_3 f_q = 2\pi\delta(k-q)\,, \tag{A11}$$

$$\int\limits_{-\infty}^{+\infty} \mathrm{d}t\,f_2^+\hat{\sigma}_3 f_1 = 2\,, \qquad \int\limits_{-\infty}^{+\infty} \mathrm{d}t\,f_0^+\hat{\sigma}_3 f_3 = -2\,. \tag{A12}$$

Let us now modify the definition of v:

$$\Psi = \left[e^{i\alpha}\cosh^{-1}(t) + v\right]\exp(iz)\,.$$

Then, the operator describing the linearized dynamics of v is

$$\hat{L}_\alpha = \left(\partial_t^2 - 1\right)\hat{\sigma}_3 + \frac{2}{\cosh^2 t}\left[2\hat{\sigma}_3 + \begin{pmatrix}0 & e^{2i\alpha}\\ -e^{-2i\alpha} & 0\end{pmatrix}\right] \tag{A13}$$

The operator $\hat{L}_\alpha$ satisfies the same identities (A4) as $\hat{L}$ does. The eigen functions of the operator (A13) can be obtained from Eqs. (A6,A7) by an obvious phase shift. One gets

$$f_{\alpha,k}(t) = \exp(i\,k\,t)\left\{1 - \frac{2\,i\,k\,\exp(-t)}{(k+i)^2\cosh(t)}\right\}\begin{pmatrix}0\\e^{-i\alpha}\end{pmatrix} + \frac{\exp(i\,k\,t)}{(k+i)^2\cosh^2(t)}\begin{pmatrix}e^{i\alpha}\\e^{-i\alpha}\end{pmatrix},$$

$$\bar{f}_{\alpha,k}(t) = \exp(-i\,k\,t)\left\{1 + \frac{2\,i\,k\,\exp(-t)}{(k-i)^2\cosh(t)}\right\}\begin{pmatrix}e^{i\alpha}\\0\end{pmatrix} + \frac{\exp(-i\,k\,t)}{(k-i)^2\cosh^2(t)}\begin{pmatrix}e^{i\alpha}\\e^{-i\alpha}\end{pmatrix} \tag{A14}$$

The eigen-functions (A14) possess the same orthogonality properties (A11) as f_k, $\bar{f}_k$ do.

APPENDIX B: INTERACTION OF TWO SOLITONS

Here we examine statistics of the force written in the right-hand side of the equation (3.10).

One starts from analyzing $\mathcal{F}_{vv}$ given by Eq. (3.11). Substituting φ_k and $\bar{\varphi}_k$ into Eqs. (1.7) one derives

$$\begin{aligned} v^2 + (v^*)^2 + 4|v|^2 = &\int \frac{\mathrm{d}q\,\mathrm{d}k}{(2\pi)^2}\frac{e^{ikx-iqx}a_k a_q^*}{(k+i)^2(q-i)^2} \times\Bigg\{\frac{2}{\cosh^2 x}\left[(q-i\tanh x)^2 + (k+i\tanh x)^2\right] \\ &+4\cosh^{-4}x + 4(q-i\tanh x)^2(k+i\tanh x)^2\Bigg\} \\ &+\int \frac{\mathrm{d}q\,\mathrm{d}k}{(2\pi)^2}\frac{e^{ikx+iqx}a_k a_q}{(k+i)^2(q+i)^2}\Bigg\{\frac{1}{\cosh^4 x} + \left[(k+i)^2 - 2ik\frac{e^{-x}}{\cosh x} + \frac{1}{\cosh^2 x}\right] \times\left[(q+i)^2 - 2iq\frac{e^{-x}}{\cosh x} + \frac{1}{\cosh^2 x}\right] \\ &+\frac{4}{\cosh^2 x}\left[(q+i)^2 - 2iq\frac{e^{-x}}{\cosh x} + \frac{1}{\cosh^2 x}\right]\Bigg\} \\ &+\int \frac{\mathrm{d}q\,\mathrm{d}k}{(2\pi)^2}\frac{e^{-ikx-iqx}a_k^* a_q^*}{(k-i)^2(q-i)^2}\Bigg\{\frac{1}{\cosh^4 x} + \left[(k-i)^2 + 2ik\frac{e^{-x}}{\cosh x} + \frac{1}{\cosh^2 x}\right] \times\left[(q-i)^2 + 2iq\frac{e^{-x}}{\cosh x} + \frac{1}{\cosh^2 x}\right] \\ &+\frac{4}{\cosh^2 x}\left[(q-i)^2 + 2iq\frac{e^{-x}}{\cosh x} + \frac{1}{\cosh^2 x}\right]\Bigg\}\,. \end{aligned} \tag{B1}$$

Making substitution of Eq. (B1) into Eq. (3.11) and taking integrals over x, one finds

$$\begin{aligned} \mathcal{F}_{vv} = &\int \frac{\mathrm{d}k\,\mathrm{d}q}{24\pi}\frac{ia_k a_q^*(k^2-q^2)^2(1+k^2+kq+q^2)}{(k+i)^2(q-i)^2\sinh[\pi(k-q)/2]} \\ &+\int \frac{\mathrm{d}k\,\mathrm{d}q}{2\pi}\frac{ia_k a_q(k+q)^2(1+k^2+q^2-kq)(2+k^2+q^2)}{24(k+i)^2(q+i)^2\sinh[\pi(k+q)/2]} \\ &-\int \frac{\mathrm{d}k\,\mathrm{d}q}{2\pi}\frac{ia_k^* a_q^*(k+q)^2(1+k^2+q^2-kq)(2+k^2+q^2)}{24(k-i)^2(q-i)^2\sinh[\pi(k+q)/2]}\,. \end{aligned} \tag{B2}$$

From Eq. (1.13,3.6,3.7) and Eq. (B2), one derives $\mathcal{F}_{vv} =$

$F+\Phi+\Phi^*$, where the quantities F and Φ are defined as

$$\begin{aligned} F &= \frac{\pi i}{3\cdot 2^5}\int \frac{\mathrm{d}k\,\mathrm{d}q\,(k^2-q^2)^2(1+k^2+q^2+kq)}{\cosh[\pi k/2]\cosh[\pi q/2]\sinh[\pi(k-q)/2]} \\ &\times \int_0^z \mathrm{d}z_1\mathrm{d}z_2\,\xi(z_1)\xi(z_2)e^{i(k^2+1)(z-z_1)-i(q^2+1)(z-z_2)} \\ &\times \left[\left(\frac{k-i}{k+i}\right)^2\left(\frac{q+i}{q-i}\right)^2 e^{i(q-k)y}\right. \qquad \text{(B3)} \\ &+ \left.\left(\frac{k-i}{k+i}\right)^2 e^{-iky-i\alpha}+\left(\frac{q+i}{q-i}\right)^2 e^{iqy+i\alpha}\right], \\ \Phi &= -\frac{\pi i}{3\cdot 2^6}\int \frac{\mathrm{d}k\mathrm{d}q(k+q)^2(1+k^2+q^2-kq)(2+k^2+q^2)}{\cosh(\pi k/2)\cosh(\pi q/2)\sinh[\pi(k+q)/2]} \\ &\times \int_0^z \mathrm{d}z_1\mathrm{d}z_2\,\xi(z_1)\xi(z_2)e^{i(k^2+1)(z-z_1)+i(q^2+1)(z-z_2)} \\ &\times \left[\left(\frac{k-i}{k+i}\right)^2\left(\frac{q-i}{q+i}\right)^2 e^{-i(q+k)y-2i\alpha}\right. \qquad \text{(B4)} \\ &+ \left.\left(\frac{k-i}{k+i}\right)^2 e^{-iky-i\alpha}+\left(\frac{q-i}{q+i}\right)^2 e^{-iqy-i\alpha}\right]. \end{aligned}$$

The second term in the force (3.12) can be analogously presented as

$$\begin{aligned} \mathcal{F}_{\xi v} &= \frac{\pi}{60}\xi(z)\int_0^z \mathrm{d}z'\xi(z')\int \frac{\mathrm{d}k\,k(k^2+1)(16+9k^2)}{\cosh^2(\pi k/2)} \\ &\times \mathrm{Re}\left[e^{i(k^2+1)(z-z')}\left(\frac{k-i}{k+i}\right)^2 e^{-i\alpha-iky}\right]. \qquad \text{(B5)} \end{aligned}$$

The third term, originating from the phase α_1 dependence on z (in the leading first order over ξ, see Eq. (3.8)), in the force (3.13) is given by

$$\begin{aligned} \mathcal{F}_{\xi\alpha} &= \frac{\xi(z)}{2\pi}\int \mathrm{d}k\mathrm{Re}\left[\int \frac{dx\tanh x}{\cosh x}a_k(z)(f_k^{(1)}+f_k^{(2)})\right] \\ &= \frac{\xi(z)}{2\pi}\int \mathrm{d}k\mathrm{Re}\left[a_k(z)\int \frac{dx e^{ikx}\tanh x}{\cosh x}\right. \\ &\times \left.\left(1-\frac{2ike^{-x}}{(k+i)^2\cosh x}+\frac{2}{(k+i)^2\cosh^2 x}\right)\right] \\ &= \frac{\xi(z)}{6}\int \frac{\mathrm{d}k\,k}{\cosh(\pi k/2)}\mathrm{Re}\frac{ia_k(z)(k-i)}{(k+i)} \\ &= -\frac{\pi}{12}\xi(z)\int_0^z \mathrm{d}z'\xi(z')\int \frac{\mathrm{d}k\,k(k^2+1)}{\cosh^2(\pi k/2)} \\ &\times \mathrm{Re}\left[e^{i(k^2+1)(z-z')}\left(\frac{k-i}{k+i}\right)^2 e^{-i\alpha-iky}\right]. \qquad \text{(B6)} \end{aligned}$$

The expressions (B3,B4,B5,B6) will be used below to examine statistics of the overall force $\mathcal{F}_{vv}+\mathcal{F}_{\xi v}+\mathcal{F}_{\xi\alpha}$ acting on the soliton.

a. *Alternative Representation*

The overall force can also be presented as

$$\mathcal{F}_{vv}+\mathcal{F}_{\xi v}+\mathcal{F}_{\xi\alpha}=\partial_z\left(\tilde{P}+P+P^*\right)+\Lambda, \qquad \text{(B7)}$$

$$\begin{aligned} \tilde{P} &= \frac{\pi}{3\cdot 2^5}\int \frac{\mathrm{d}k\,\mathrm{d}q\,(k^2-q^2)(1+k^2+q^2+kq)}{\cosh[\pi k/2]\cosh[\pi q/2]\sinh[\pi(k-q)/2]} \\ &\times \int_0^z \mathrm{d}z_1\mathrm{d}z_2\,\xi(z_1)\xi(z_2)e^{i(k^2+1)(z-z_1)-i(q^2+1)(z-z_2)} \\ &\times \left[\left(\frac{k-i}{k+i}\right)^2\left(\frac{q+i}{q-i}\right)^2 e^{i(q-k)y}\right. \qquad \text{(B8)} \\ &+ \left.\left(\frac{k-i}{k+i}\right)^2 e^{-iky-i\alpha}+\left(\frac{q+i}{q-i}\right)^2 e^{iqy+i\alpha}\right], \end{aligned}$$

$$\begin{aligned} P &= -\frac{\pi}{3\cdot 2^6}\int \frac{\mathrm{d}k\mathrm{d}q(k+q)^2(1+k^2+q^2-kq)}{\cosh(\pi k/2)\cosh(\pi q/2)\sinh[\pi(k+q)/2]} \\ &\times \int_0^z \mathrm{d}z_1\mathrm{d}z_2\,\xi(z_1)\xi(z_2)e^{i(k^2+1)(z-z_1)+i(q^2+1)(z-z_2)} \\ &\times \left[\left(\frac{k-i}{k+i}\right)^2\left(\frac{q-i}{q+i}\right)^2 e^{-i(q+k)y-2i\alpha}\right. \qquad \text{(B9)} \\ &+ \left.\left(\frac{k-i}{k+i}\right)^2 e^{-iky-i\alpha}+\left(\frac{q-i}{q+i}\right)^2 e^{-iqy-i\alpha}\right], \end{aligned}$$

$$\begin{aligned} \Lambda &= \frac{\pi\xi(z)}{8}\mathrm{Re}\int \frac{\mathrm{d}k\,k(1+k^2)^2}{\cosh^2(\pi k/2)} \qquad \text{(B10)} \\ &\times \int_0^z \mathrm{d}z'\,\xi(z')e^{i(k^2+1)(z-z')}\left[\left(\frac{k-i}{k+i}\right)^2 e^{-i\alpha-iky}\right], \end{aligned}$$

where the exponentially small in y terms are omitted. (The terms are produced by integrals, say, over k, with the oscillating, $\sim\exp(-iky)$ and z-independent integrands. Then, the integration contour can be shifted to surround a pole, nearest to the real axis, and a residue at the pole gives the main contribution, exponentially small over y.)

b. *Second soliton*

Straightforward calculations show that for the force acting on the second soliton one can, actually, use Eqs. (B3,B4,B5,B6,B10) with the only correction there: expressions under square brackets on the right-hand side of each of those formulas should be replaced by their complex conjugates.

1. Average Impulse

Here we calculate the average of the overall force $\mathcal{F}_{vv}+\mathcal{F}_{\xi v}$, given by (B7), over statistics of ξ. Notice, that the average of Λ, calculated in accordance with Eqs. (1.2,B10), is exponentially small in $y = y_2 - y_1$ and will be neglected below.

It follows from Eq. (B8) that

$$\begin{aligned}\left\langle \tilde{P} \right\rangle &= \frac{\pi i D}{96} \int \mathrm{d}k\, \mathrm{d}q \, \left\{1 - \exp[i(k^2-q^2)z]\right\} \\ &\times \frac{1+k^2+q^2+kq}{\cosh(\pi k/2)\cosh(\pi q/2)\sinh[\pi(k-q)/2]} \\ &\times \left[\left(\frac{k-i}{k+i}\right)^2 \left(\frac{q+i}{q-i}\right)^2 e^{i(q-k)y}\right. \\ &+ \left.\left(\frac{k-i}{k+i}\right)^2 e^{-iky-i\alpha} + \left(\frac{q+i}{q-i}\right)^2 e^{iqy+i\alpha}\right].\end{aligned} \tag{B11}$$

Let us change the integration variables from k, q to $k_\pm = k \pm q$. The first contribution to the average impulse originates from the first term inside the brackets in Eq. (B11)

$$\begin{aligned}\left\langle \tilde{P} \right\rangle_1 &= \frac{\pi i D}{192} \int \frac{\mathrm{d}k_+ \mathrm{d}k_-}{\sinh(\pi k_-/2)} \left\{1 - e^{ik_+k_-z}\right\} e^{-ik_-y} \\ &\times \frac{1+k^2+q^2+kq}{\cosh(\pi k/2)\cosh(\pi q/2)} \frac{(k-i)^2}{(k+i)^2} \frac{(q+i)^2}{(q-i)^2}.\end{aligned}$$

The integral is formed at the smallest k_-. One gets

$$\left\langle \tilde{P} \right\rangle_1 = \frac{\pi D}{48} \int_0^\infty \mathrm{d}k_+ \, \frac{1+3k_+^2/4}{\cosh^2(\pi k_+/4)} = \frac{D}{6}\,, \tag{B12}$$

where terms exponentially small in y are omitted. The second contribution to the average impulse coming from the last two terms inside the square brackets in Eq. (B11) is formed at small $k_\pm$ and can be written as

$$\begin{aligned}\left\langle \tilde{P} \right\rangle_2 &= \frac{iD}{96} \int \frac{\mathrm{d}k_-\, \mathrm{d}k_+}{k_-} \left\{1 - \exp[ik_+k_-z]\right\} \\ &\times \left[e^{-i(k_+y/2+k_-y/2+\alpha)} + e^{i(k_+y/2-k_-y/2+\alpha)}\right] \\ &= \frac{\pi D}{12y} \sin(\alpha + y^2/4z).\end{aligned}$$

One finds, that at large y the contribution given by Eq. (B12) is dominant.

Let us now consider the average

$$\begin{aligned}\langle P \rangle &= -\frac{\pi i D}{3 \cdot 2^6} \int \frac{\mathrm{d}k \mathrm{d}q (k+q)^2 (1+k^2+q^2-kq)}{\cosh(\pi k/2)\cosh(\pi q/2)} \\ &\times \frac{[1 - e^{i(k^2+1)z + i(q^2+1)z}]}{(2+k^2+q^2)\sinh[\pi(k+q)/2]} \left[\left(\frac{k-i}{k+i}\right)^2 e^{-iky-i\alpha}\right. \\ &+ \left.\left(\frac{q-i}{q+i}\right)^2 e^{-iqy-i\alpha} + \left(\frac{k-i}{k+i}\right)^2 \left(\frac{q-i}{q+i}\right)^2 e^{-i(q+k)y-2i\alpha}\right].\end{aligned}$$

The term, which does not contain a z-dependence, produce an exponentially subleading in y contribution to $\langle P \rangle$. The z-dependent contribution is formed $q, k \sim 1/\sqrt{z}$, and it is, therefore, $\sim y/z^2$. (Notice also, that the term oscillates rapidly with z also.) Therefore, the averages $\langle P \rangle$, $\langle P^* \rangle$ are negligible at large z in comparison with the contribution given by Eq. (B12).

To conclude, at large z the average force is zero and the main contribution to the impulse of the force $\mathcal{F}$ is $D/6$.

2. Fluctuations of the Force

One considers here the irreducible part of the pair correlation of $\mathcal{F}_{vv}$, which can be written as

$$\begin{aligned}\langle\!\langle \mathcal{F}_{vv}(z_1)\mathcal{F}_{vv}(z_2) \rangle\!\rangle &= \langle \mathcal{F}_1 \mathcal{F}_2 \rangle - \langle \mathcal{F}_1 \rangle \langle \mathcal{F}_2 \rangle \\ &= \langle F_1 F_2 \rangle + \langle \Phi_1 \Phi_2^* \rangle + \langle \Phi_1^* \Phi_2 \rangle\,,\end{aligned} \tag{B13}$$

where $\langle \mathcal{F}_{vv} \rangle$ is neglected and only non-oscillating terms are kept.

The first contribution to Eq. (B13) is

$$\begin{aligned}\langle F_1 F_2 \rangle - \langle F_1 \rangle \langle F_2 \rangle &= \frac{\pi^2 D^2}{9 \cdot 2^{10}} \\ \times \int & \frac{\mathrm{d}k_1 \mathrm{d}k_2 \mathrm{d}q_1 \mathrm{d}q_2 \, (k_1^2-q_1^2)^2 (k_2^2-q_2^2)^2}{\cosh[\pi k_1/2]\cosh[\pi q_1/2]\cosh[\pi k_2/2]\cosh[\pi q_2/2]} \\ &\times \frac{\left(1+k_1^2+q_1^2+k_1q_1\right)\left(1+k_2^2+q_2^2+k_2q_2\right)}{\sinh[\pi(k_1-q_1)/2]\sinh[\pi(k_2-q_2)/2]} e^{i(k_1^2-q_1^2)\zeta} \\ &\times \left[\left(\frac{k_1-i}{k_1+i}\right)^2 e^{-iq_1y-i\alpha} + \left(\frac{q_1+i}{q_1-i}\right)^2 e^{ik_1y+i\alpha}\right. \\ &+ \left.\left(\frac{k_1-i}{k_1+i}\right)^2 \left(\frac{q_1+i}{q_1-i}\right)^2\right] \left[\left(\frac{k_2-i}{k_2+i}\right)^2 \left(\frac{q_2+i}{q_2-i}\right)^2\right. \\ &+ \left.\left(\frac{k_2-i}{k_2+i}\right)^2 e^{-iq_2y-i\alpha} + \left(\frac{q_2+i}{q_2-i}\right)^2 e^{ik_2y+i\alpha}\right] \\ &\times \frac{1}{k_+k_-q_+q_-} \left[e^{ik_+k_-z} - 1\right] \left[e^{iq_+q_-z} - 1\right] e^{-ik_-y-iq_-y}.\end{aligned} \tag{B14}$$

where $k_\pm = k_1 \pm q_2$, $q_\pm = k_2 \pm q_1$, and $z = \min\{z_1, z_2\}$, $\zeta = |z_1 - z_2|$. The simultaneous correlation function, correspondent to $\zeta = 0$, is the first object to study here. One finds that the dominant contribution, proportional to logarithm of y and z, originates from the α-independent terms in the integrand of (B14). (The α-dependent contribution is $\sim 1/y$.) There are such contributions of two kinds. The first one comes from the product of two different α-independent terms, each from expression bounded by the square brackets in the integrand of Eq. (B14). The terms of the second kind are coming from products of two terms cancelling their α-dependence in the result. In the contribution of the first kind, the integrals over k_- and q_- are formed at both $k_-, q_- \sim 1/y$. Thus, replacing k_-, q_- in all non-oscillatory terms by zero, one derives

$$\begin{aligned}\langle\!\langle F^2 \rangle\!\rangle_1 &= \frac{\pi^4 D^2}{9 \cdot 2^{18}} \int_{y/z}^\infty \frac{\mathrm{d}k_+ \mathrm{d}q_+ \, (k_+^2 - q_+^2)^4}{\cosh^2[\pi k_+/4]\cosh^2[\pi q_+/4]} \\ &\times \frac{(1 + k_+^2/4 + q_+^2/4 + k_+q_+/4)^2}{k_+q_+ \sinh^2[\pi(k_+ - q_+)/4]}.\end{aligned} \tag{B15}$$

Some products produce analogous integrals (originating from small k_- and q_-) containing, however, the factors oscillating with k_+ and/or q_+ in the integrands. The contributions are $\propto y^{-1}$ and can be drooped. In the remaining two (identical) contributions of the second type the k_- and q_- integration are not equivalent. One of the wave-vectors, say k_- is still $O(1/y)$. Integrating over k_- one gets

$$\langle\langle F^2\rangle\rangle_2 = \frac{\pi^4 D^2}{9\cdot 2^{21}}\int_{y/z}^{\infty}\frac{\mathrm{d}q_+}{q_+\cosh^2(\pi q_+/4)} \tag{B16}$$
$$\times\int\frac{\mathrm{d}k_+\mathrm{d}k_-}{2\pi i k_+k_-}\left[\exp(ik_+k_-z)-1\right]$$
$$\times\frac{[(k_++k_-)^2-q_+^2]^2[(k_+-k_-)^2-q_+^2]^2}{\cosh[\pi(k_++k_-)/4]\cosh[\pi(k_+-k_-)/4]}$$
$$\times\frac{[4+(k_++k_-)^2+q_+^2+q_+(k_++k_-)][k_-\to -k_-]}{\sinh[\pi(k_++k_--q_+)/4]\sinh[\pi(k_+-k_--q_+)/4]}.$$

The major contribution into the integral originates from $q_+\sim 1\gg k_\pm$. Replacing the integrand in Eq. (B16) by its asymptotic value at $k_\pm\to 0$, one finds that integration over $k_\pm$ gives a $\sim\ln[z]$ contribution. Finally, collecting the two major contribution into the simultaneous correlation function one finds

$$\langle\langle F^2\rangle\rangle = \frac{\pi^4 D^2}{9\cdot 2^{15}}[\ln(z/y)+\ln z]\int_0^{\infty}\frac{\mathrm{d}q\, q^7(1+q^2/4)^2}{\sinh^2(\pi q/2)}$$
$$= 0.0068\cdot D^2\cdot\ln[z^2/y]\,. \tag{B17}$$

The result (B17) is asymptotic, valid at $z\gg y$ only. Let us now account for $\zeta\neq 0$ in Eq. (B14), i.e. for $z_1\neq z_2$. It is obvious from the analysis of the simultaneous correlation function that Eq. (B17) is formed at such values of the four wave vectors $k_\pm$, $q_\pm$, that only one of the wave vectors is $O(1)$, while the other three much smaller. The modification of Eq. (B17) is

$$\langle\langle F(z_1)F(z_2)\rangle\rangle = \frac{\pi^4 D^2}{9\cdot 2^{15}}\ln(z^2/y)$$
$$\times\int_0^{\infty}\frac{\mathrm{d}q\, q^7(1+q^2/4)^2}{\sinh^2(\pi q/2)}\cos\left[q^2\zeta\right], \tag{B18}$$

i.e. at $\zeta\gg 1$, the correlations decay algebraically in ζ, $\langle\langle F(z_1)F(z_2)\rangle\rangle\sim D^2\ln[z^2/y]/\zeta^3$. It also follows from Eq. (B18) that $\int\mathrm{d}\zeta\,\langle\langle F(z+\zeta)F(z)\rangle\rangle = 0$.

Let us calculate the Φ and Φ^* in Eq. (B13). One finds

$$\langle\Phi(z+\zeta/2)\Phi^*(z-\zeta/2)\rangle = -\frac{\pi^2 D^2}{9\cdot 2^{11}} \tag{B19}$$
$$\times\int\frac{\mathrm{d}k_1\mathrm{d}q_1(k_1+q_1)^2(1+k_1^2+q_1^2-k_1q_1)(2+k_1^2+q_1^2)}{\cosh(\pi k_1/2)\cosh(\pi q_1/2)\sinh[\pi(k_1+q_1)/2]}$$
$$\times\int\frac{\mathrm{d}k_2\mathrm{d}q_2(k_2+q_2)^2(1+k_2^2+q_2^2-k_2q_2)(2+k_2^2+q_2^2)}{\cosh(\pi k_2/2)\cosh(\pi q_2/2)\sinh[\pi(k_2+q_2)/2]}$$
$$\times\frac{e^{i(k_1^2/2+k_2^2/2+1)\zeta}}{k_1^2-k_2^2}\left[e^{i(k_1^2-k_2^2)|\zeta|/2}-e^{i(k_1^2-k_2^2)z}\right]$$
$$\times\frac{e^{i(q_1^2/2+q_2^2/2+1)\zeta}}{q_1^2-q_2^2}\left[e^{i(q_1^2-q_2^2)|\zeta|/2}-e^{i(q_1^2-q_2^2)z}\right]$$
$$\times\left[\left(\frac{k_1-i}{k_1+i}\right)^2\left(\frac{q_1-i}{q_1+i}\right)^2 e^{-i(q_1+k_1)y-2i\alpha}\right.$$
$$\left.+\left(\frac{k_1-i}{k_1+i}\right)^2 e^{-ik_1y-i\alpha}+\left(\frac{q_1-i}{q_1+i}\right)^2 e^{-iq_1y-i\alpha}\right]$$
$$\times\left[\left(\frac{k_2+i}{k_2-i}\right)^2\left(\frac{q_2+i}{q_2-i}\right)^2 e^{i(q_2+k_2)y+2i\alpha}\right.$$
$$\left.+\left(\frac{k_2+i}{k_2-i}\right)^2 e^{ik_2y+i\alpha}+\left(\frac{q_2+i}{q_2-i}\right)^2 e^{iq_2y+i\alpha}\right].$$

The first logarithmic contribution to the average (B19) originates from the terms, containing 2α. The integrals are formed at small values of k_1-k_2 and q_1-q_2. The result of integration is

$$\langle\Phi(z+\zeta/2)\Phi^*(z-\zeta/2)\rangle_1 = \frac{\pi^4 D^2}{9\cdot 2^{15}}\int_{y/z}^{\infty}\frac{\mathrm{d}k_+\mathrm{d}q_+}{k_+q_+}$$
$$\frac{(1+k_+^2/4+q_+^2/4-k_+q_+/4)^2(2+k_+^2/4+q_+^2/4)^2}{\cosh^2(\pi k_+/4)\cosh^2(\pi q_+/4)\sinh^2[\pi(k_++q_+)/4]}$$
$$\times(k_++q_+)^4\exp\left[i(k_+^2/4+q_+^2/4+2)\zeta\right], \tag{B20}$$

where $q_+=q_1+q_2$ and $k_+=k_1+k_2$. If $\zeta=0$ then the integral (B20) is reduced to

$$\langle\Phi(z)\Phi^*(z)\rangle_1 = \frac{\pi^4 D^2}{9\cdot 2^{12}}\ln(z/y) \tag{B21}$$
$$\times\int_0^{\infty}\mathrm{d}k_+\frac{k_+^3(1+k_+^2/4)^2(2+k_+^2/4)^2}{\sinh^2[\pi k_+/2]}\approx 0.017D^2\ln(z/y)\,.$$

The integral of the expression (B20) over ζ is evidently zero.

We now turn to calculation of the second logarithmic correction originating from the terms which contain α. The contribution is

$$\langle\Phi(z+\zeta/2)\Phi^*(z-\zeta/2)\rangle_2 = \frac{\pi^4 D^2}{9\cdot 2^{12}} \tag{B22}$$
$$\times\int\frac{\mathrm{d}q_+\,q_+^3(1+q_+/4)^2(2+q_+^2/4)^2}{\sinh^2[\pi q_+/2]}e^{i(q_+^2/4+2)\zeta}$$
$$\times\int\frac{\mathrm{d}k_+\mathrm{d}k_-}{2\pi i k_+k_-}\left[e^{ik_+k_-z}-1\right]$$

where $k_\pm \ll q_+$, as in Eq. (B16). The integral over k_+ and k_- in Eq. (B22) produces $\ln z$. Therefore, the term $\langle\Phi(z)\Phi^*(z)\rangle_2$ is given by expression similar to Eq. (B21), with the substitution of $\ln(z/y)$ by $\ln z$. Finally,

$$\langle\Phi(z)\Phi^*(z)\rangle \approx 0.017D^2\ln(z^2/y)\,, \tag{B23}$$

$$\int \mathrm{d}z\,\langle\Phi(z+\zeta/2)\Phi^*(z-\zeta/2)\rangle = 0\,. \tag{B24}$$

a. *Dominant (Λ-) term*

The pair correlation function of the contribution (B10) is

$$\langle\Lambda(z)\Lambda(z')\rangle = D^2 G\delta(z-z'), \tag{B25}$$

$$G = \frac{\pi^2}{2^7}\int \mathrm{d}k\mathrm{d}q\,\frac{e^{i(k^2-q^2)z}-1}{i(k^2-q^2)} \times\frac{kq(1+k^2)^2(1+q^2)^2}{\cosh^2(\pi k/2)\cosh^2(\pi q/2)} \times\left(\frac{k-i}{k+i}\right)^2\left(\frac{q+i}{q-i}\right)^2 e^{i(q-k)y}\,. \tag{B26}$$

The major, y and z-independent contribution into G is coming from small values of $k-q$. Taking the integral over this variable, one derives

$$G=\frac{\pi^3}{2^7}\int_0^\infty\frac{\mathrm{d}k k(1+k^2)^4}{\cosh^4(\pi k/2)}\approx 0.14. \tag{B27}$$

Correlation function which defines change in the relative position of the two solitons is defined by $\tilde\Lambda = \Lambda_1-\Lambda_2$, where the indexes (1) and (2) stand for the first and second solitons, i.e. Λ_1 is given by Eq. (B10), while Λ_2 is given by Eq. (B10) with the expression on the right-hand side of it replaced by its complex conjugate. One gets

$$\langle\tilde\Lambda(z)\tilde\Lambda(z')\rangle = 2[1+\cos(2\alpha)]D^2 G\delta(z-z'). \tag{B28}$$

3. Fluctuations of the Impulse

Similarly to the calculations of the previous Subsection one can analyse fluctuations of the impulse $\mathcal{P} = \tilde P + P + P^*$. We obtain instead of Eq. (B17)

$$\langle\!\langle\tilde P^2\rangle\!\rangle = \frac{\pi^4 D^2}{9\cdot 2^{11}}[\ln(z/y)+\ln z]\int_0^\infty\frac{\mathrm{d}q\,q^3(1+q^2/4)^2}{\sinh^2(\pi q/2)} \approx 0.0036D^2\ln(z^2/y)\,. \tag{B29}$$

The analog of Eq. (B21) is

$$\langle P(z)P^*(z)\rangle = \frac{\pi^4 D^2}{9\cdot 2^{12}}[\ln(z/y)+\ln z] \times\int_0^\infty \mathrm{d}k_+\frac{k_+^3(1+k_+^2/4)^2}{\sinh^2[\pi k_+/2]} \approx 0.0018D^2\ln(z^2/y)\,. \tag{B30}$$

Finally, one gets the following answer for the pair simultaneous correlation function of the impulse

$$\left\langle\mathcal{P}^2(z)\right\rangle \approx 0.0073D^2\ln(z^2/y)\,. \tag{B31}$$

The major contribution into the overall impulse of the force is coming from the Λ-term

$$\left\langle\left[\int_0^z \mathrm{d}z'\,\Lambda(z')\right]^2\right\rangle \approx 0.265D^2 z\,. \tag{B32}$$

The cross correlations are given by

$$\langle\mathcal{P}(z+\zeta)\Lambda(z)\rangle = \frac{\pi^2D^2}{9\cdot 2^7}\mathrm{Re}\int \mathrm{d}k\,\mathrm{d}q\,\mathrm{d}p\,e^{i(k^2-q^2)\zeta} \tag{B33}$$
$$\times\frac{k(k-q)(1+k^2+q^2+kq)}{\cosh[\pi k/2]\cosh[\pi q/2]\sinh[\pi(k-q)/2]}\left(\frac{q+i}{q-i}\right)^2 \times\frac{p(1+p^2)(5+3p^2)}{\cosh^2(\pi p/2)}\left(\frac{p-i}{p+i}\right)^2\frac{e^{i(p^2-q^2)z}-1}{i(p^2-q^2)}e^{i(q-p)y}\,,$$

which is non-zero at $\zeta>0$ only. The integral (B33) is formed at the small values of $p-q$. Integrating one finds

$$\langle\mathcal{P}(z+\zeta)\Lambda(z)\rangle = -\frac{\pi^3D^2}{9\cdot 2^7}\mathrm{Re}\int \mathrm{d}k\int_0^\infty \mathrm{d}q\,e^{i(k^2-q^2)\zeta} \times\frac{k(k-q)(1+k^2+q^2+kq)(1+q^2)(5+3q^2)}{\cosh[\pi k/2]\cosh^3[\pi q/2]\sinh[\pi(k-q)/2]}\,, \tag{B34}$$

which turns to the following expression at $\zeta\to 0$

$$\langle\mathcal{P}(z+0)\Lambda(z)\rangle = -\frac{\pi^3D^2}{9\cdot 5\cdot 2^7}\int_0^\infty \mathrm{d}q \times\frac{q(1+q^2)^2(5+3q^2)(7+3q^2)}{\cosh^4[\pi q/2]} \approx -0.068D^2\,. \tag{B35}$$

We also find from Eq. (B34)

$$\left\langle\mathcal{P}(z)\int_0^z \mathrm{d}z'\Lambda(z')\right\rangle = -\frac{\pi^3D^2}{9\cdot 2^7}\mathrm{Re}\int \mathrm{d}k\,k\int_0^\infty \mathrm{d}q \frac{e^{i(k^2-q^2)z}-1}{i(k^2-q^2)}\,\frac{(k-q)(1+k^2+q^2+kq)(1+q^2)(5+3q^2)}{\cosh[\pi k/2]\cosh^3[\pi q/2]\sinh[\pi(k-q)/2]}$$
$$= -\frac{\pi^3D^2}{9\cdot 2^7}\left\{\int_0^\infty \mathrm{d}q\frac{(1+3q^2)(1+q^2)(5+3q^2)}{\cosh^4(\pi q/2)} -\pi\int_0^\infty \mathrm{d}q\frac{q(1+q^2)^2(5+3q^2)}{\cosh^4(\pi q/2)\sinh(\pi q)}\right\} \approx -0.053D^2\,. \tag{B36}$$

Therefore this cross correlation is negligible.

4. Additional Impulse

An additional impulse, $\mathcal{P}_1$, the last one left to be calculated, is due to the direct noise contribution $\mathcal{P}_1$ in Eq.

(3.14). Expressing v, v^* in Eq. (3.15) via ξ and performing averaging over the statistics of ξ in accordance with Eq. (1.2) one finds

$$\langle \mathcal{P}_1(z)\rangle = -\frac{D}{4}\mathrm{Im}\int \frac{\mathrm{d}x\, x}{\cosh^4 x}\frac{\mathrm{d}k\,\mathrm{d}q\, e^{ikx-iqx}}{\cosh(\pi k/2)\cosh(\pi q/2)}$$
$$\frac{\exp[i(k^2-q^2)z]-1}{i(k^2-q^2)}\left\{(q-i)^2+\frac{2iqe^{-x}}{\cosh x}+\frac{1}{\cosh^2 x}\right\}$$
$$\left[1+\left(\frac{k-i}{k+i}\right)^2 e^{-iky-i\alpha}\right]\left[1+\left(\frac{q+i}{q-i}\right)^2 e^{iqy+i\alpha}\right].$$

Integrating the resulting expression over x, one derives

$$\langle \mathcal{P}_1(z)\rangle = \frac{\pi D}{192}\int \frac{\mathrm{d}k\,\mathrm{d}q\,(k-q)^2(k+q)}{\cosh(\pi k/2)\cosh(\pi q/2)}\frac{1}{i(k^2-q^2)}$$
$$\times\left[1+\left(\frac{k-i}{k+i}\right)^2 e^{-iky-i\alpha}\right]\left[1+\left(\frac{q+i}{q-i}\right)^2 e^{iqy+i\alpha}\right]$$
$$\times\frac{\exp[i(k^2-q^2)z]-1}{\sinh^2[\pi(k-q)/2]}\left\{-4(k-q)\sinh\left[\frac{\pi(k-q)}{2}\right]\right.$$
$$\left.+\pi(4+k^2-2kq+q^2)\cosh\left[\frac{\pi(k-q)}{2}\right]\right\}. \quad \text{(B37)}$$

The main contribution into the integral comes from k close to q. Simplifying the expression, keeping only main terms in $k-q$, one can then take the integrals over k and q, obtaining $\langle \mathcal{P}_1(z)\rangle = D/3$. The contribution should be taken into account on equal footing with Eq. (B12). That gives a systematic drift $2Dz/3$ for y_1.

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